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Theorizing Mathematical Proof as Becoming: A Deleuzio-Guattarian Investigation

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ABSTRACT

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In this dissertation, I utilize the post-structural philosophy of Gilles Deleuze and Félix Guattari as a lens for investigating the proof process. Deleuze and Guattari were both post-structural philosophers who, like many in this tradition, troubled traditional notions related to stable identities, meaning, language, and mathematics. For Deleuze, sense and meaning is not the result of a sterile, transcendent effect or condition that is associated with propositions and states of affairs. Rather, it is the result of a material production that emerges from the world and that has independence from language and the mind. I apply this framework to the notion of proof, situating the work of the mathematician not in terms of the problem-solving subject, cognition, and mental processes but rather as something that emerges from constructed material assemblages.

I demonstrate this theoretical approach via a story written by Dr. David Neel from the 2019 book *Living Proof: Stories of Resilience Along the Mathematical Journey* published by the American Mathematical Society. Neel's story is an autobiographical account that takes place near the end of his dissertation work in which both he and his supervisor go hiking while working out the details regarding a challenging result that is related to his research. This story illustrates that, while proof involves logical reasoning and mental processes, these aspects are only part of the construction. The axiomatic proof which they try to obtain becomes entangled with the hike and, at the level of the assemblage, what they are constructing is a "proof-hike" that serves as the genesis for the production of an actualized proof solution. Not only does this "proof-hike" eventually lead to a mathematical resolution but it also leads to a sort of ethical disposition in which Neel (2019) sees the axiomatic aspect of mathematics as only a part of an entire universe that opens up to him. The result is an intensity, an "affect," that is both compassionate as well as embracing of the "cosmic" wonder of proof at its material genesis on the plane of immanence. It is a becoming-proof.

Drawing on the literature, I end the dissertation by formulating potential possibilities for how this approach might affect our thinking about the teaching and learning of proof. For example, I suggest approaching proof from a problematic standpoint, as opposed to an axiomatic one, where the role of exploration and wonder is foregrounded rather than solutions that lead to theorems. These approaches, I argue, have the potential to deterritorialize proof and proof instruction so that those who may not feel connected to the axiomatic side of proof can still desire it immanently.

Dedication

I dedicate this dissertation to my Mom, Dad, sister and brother.

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Introduction

Proof, that is, the process of deducing new mathematical theorems, is usually seen as a highly signifying phenomenon. For example, Rota (1997) describes a proof as "a sequence of steps which leads to the desired conclusions" (p. 183) while Nathanson (2009) describes proof as something that "typically consists of a series of assertions, each leading more or less to the next, and concluding in the statement of the theorem" (p. 8). Both of these accounts point to "conclusions" that are "the statement of the theorem" to be proved. For example, consider the statement: *The sum of two even integers is also even*. How do I arrive at such a conclusion? Via a deductive process that, in this case, emphases algebraic manipulations: If I let p and q *be* two arbitrary even integers and recall that an even integer can be expressed as a multiple of 2 (definition of even integer), I have that $p = 2n$ and $q = 2m$ for some integers n and m. This implies that $p + q = 2n + 2m = 2(n + m)$. Therefore, $p + q$ is an even integer and the statement is obtained. I will refer to this sort of approach to proof as *axiomatic* since I begin with premises (and their underlying axioms) and end with a theorem. Axiomatic proof signifies the theorems and facts that constitute mathematics itself.

While the above approach to proof foregrounds the often rigorous nature of this process, there are other ways to conceptualize mathematical argumentation that foreground the meaning and the more explanatory aspects of proof. For example, Stylianou et al. (2015) provides the following "deductive narrative" justification concerning the theorem from the previous paragraph:

> Even numbers are numbers that can be divided by *2*. When you add numbers with a common factor, 2 in this case, the answer will have the same common factor. Therefore, the sum of any two even numbers is always even. (p. 98)

While the above proof may not be considered as algebraically rigorous as the initial justification, it still provides a fairly clear meaning, particularly regarding the role of 2 as a common factor. Stylianou et al. (2015) also consider the use of pictures, or "empirical-visual" justifications involving pairing where two groups of objects are presented and each of the two groups consist of an even number of said objects. Since each group consists of an even number of objects, I can pair up all objects in each group with no remainder. Therefore, the sum of the objects will also result in a larger group with each object paired and with no remainder. While this proof is often not as valued in more expert contexts, since it is not as rigorous, it perhaps more clearly illustrates the *meaning* of the original algebraic proof by focusing on object pairing as a stand in for 2 as a common factor.

Much of the mathematics education literature concerning proof attempts to demonstrate the more complex nature of proof in this way. Proof is more than just a way to rigorously verify a mathematical fact. It is also meant to elicit understanding and to function as an explanation as to why a theorem holds (e.g., Hanna, 2000; Lockwood et al., 2020). In a comprehensive review of the literature, Stylianides et al. (2017) indicate that proof as problem-solving, proof as conviction, and proof as a social activity are all important aspects of the proof literature. Weber (2001) explores the use of strategic knowledge by more advanced mathematics students while Harel and Sowder (1998) and Raman (2003) investigate the role of schemes and key ideas respectively. Other work surrounding this topic has emphasized the role of proof in secondary instruction (Knuth, 2002) and the way expert mathematicians read proof (Mejia-Ramos & Weber, 2014).

As described above, the study of mathematics education has demonstrated that proof is an important and complex activity that must be studied from a variety of perspectives. However,

much of this work appears to assume, even if only implicitly, that proof is ultimately a *signifying* phenomenon. These studies may suggest that proof should be seen in a nuanced manner and as more than simply a way to verify the truth of mathematical statements, yet the signifying relation still stands: proof, no matter how rigorous, explanatory, and mathematically meaningful, ultimately "points" to the transcendent, generalizable, and platonist world of the ideal. That is, via deductive processes, proof is often "about" the theorems, facts, and states of affairs that constitute mathematics itself. Rota (1997), however, begins to trouble this notion of signification in the context of Wiles' proof of Fermat's last theorem:

> The error lies in assuming that a mathematical proof has been devised for the explicit purpose of proving what it purports to prove. Again, appearances are deceptive. The actual value of what Wiles and his collaborators did is far greater than the mere proof of a whimsical conjecture. The point of the proof of Fermat's last theorem is to open up new possibilities for mathematics. (p. 190)

This idea that proofs are "deceptive" begins to approach the manner in which I see proof in this dissertation. It suggests that there is a function to proof that is not readily visible and perhaps never will be. Proof has a function beyond merely explaining and understanding the facts of mathematics. Its function "is to open up new possibilities." Rota (1997) writes "We propose instead that a rigorous version of the notion of possibility be added to the formal baggage of metamathematics. One cannot pretend to disregard possibility, by arguing that the possibilities of a mathematical result lie concealed beneath formal statements" (p. 191). I see the work of this dissertation as an extension of this agenda.

Concerning the above discussion regarding the signifying nature of proof, I ask some initial motivating questions: Does mathematical proof have an existence *beyond* this signifying

relation? That is, does proof function in a manner that is asignifying and in a manner that is not always in primary relation to facts and theorems? If so, what does this existence look like? What value does it have for mathematicians and for the mathematics student? Finally, how does this way of thinking about proof inform the way we as instructors approach the teaching and learning of mathematics? In this dissertation, I argue that proof as a signifying relation privileges reasoning, logic, and cognitive constructions. On the other hand, proof as an asignifying relation privileges the body, affect, and becoming (see Figure 1). The asignifying relation does not resemble the signifying relation. In the latter case, we can verify proof and encapsulate new theorems but we ultimately are unable to speak of the *becoming* of proof in the first place. On the other hand, the asignifying relation is difficult to encapsulate. It cannot offer concrete mathematical verification like the signifying relation can. However, it offers something that, in many ways, is even more powerful than mathematical results: it allows for insight regarding how proof emerges in the first place from a kind of invisible immanence. This immanence allows for the construction of an *assemblage*, that ultimately actualizes a proof or allows for the possibility of proof.

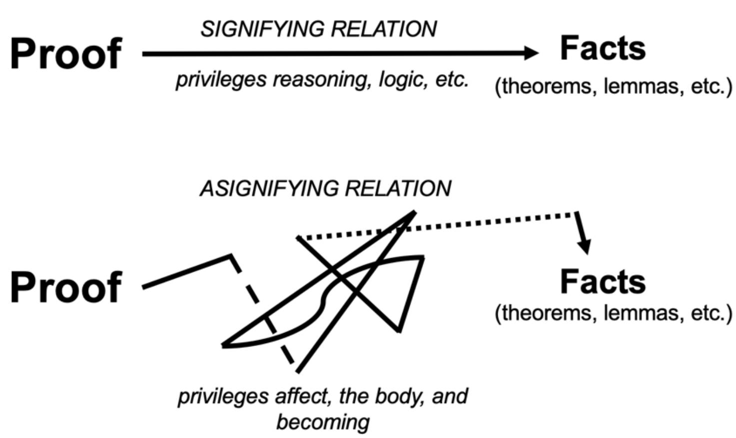

Figure 1. Proof as signifying and asignifying relation.

Exploring these issues pertaining to these relations is the main purpose of this dissertation and in this first introductory chapter, I briefly introduce the theoretical and methodological tools I will utilize to carry out this investigation and also provide brief summaries for each of the three main manuscripts of this project.

The notion of assemblage, which is a kind of construction in which affective intensities circulate and cohere, offer ways to approach the world without reducing thought to a platonic stable identity. In the case of proof, this involves seeing mathematical argumentation not as a constructive, mental process like in the case of Piaget or Vygotsky, but rather as a bringing together of material forces that somehow has the ability to communicate and flow with axiomatic elements. For example, I analyze a story written by Neel (2019) that demonstrates that proof can assemble with other physical activities, such as a hike. This "proof-hike" is a construction that builds or produces the proof event, an idea that provides the sense for a completed proof argument. In this manner, it is not the individual or self that actualizes proof, but rather the assemblage that produces it.

This production involves a bodily, felt sense of nearness to "something." Perhaps this "something" is a solution, but that is not the only aspect that provides this sense. Rather, this feeling of nearness corresponds to a movement away from the transcendent world and toward the plane of immanence in which mathematics is no longer axiomatic but intensive and affective. It is this asignifying, intensive knowledge that constitutes proof at its genesis in production rather than in the signifying reality of reason, logic, and mental processing. By forming proof assemblages, this "something" that is felt is the closeness to the plane of immanence rather than a solution to a problem. This is the perspective of proof that I take in this dissertation: proof is an immanent activity that engages in intensive, dynamic, bodily "feeling" or becoming that is

facilitated by the formation of asignifying material assemblages where these affects move and circulate. This is much different than psychological constructivist notions, such as the traditions of Piaget and Vygotsky, in which individuals' signifying, mental processes lead statically to a solution.

Gilles Deleuze and Félix Guattari: A Philosophy of Immanence

Gilles Deleuze was a French philosopher and professor of philosophy who often worked with Félix Guattari, a French psychoanalyst. Deleuze and Guattari's (1972/1983) first collaborative effort, *Anti-Oedipus*, attempted to conceptualize the notion of desire not in terms of a Freudian "lack" (i.e., the longing for an object that is yet to be present) but, rather, as a pure proliferation of affect without any connection to an object of desire (present or not). In relation to mathematical proof, the Freudian aspect can be seen in proof's signifying function: the proof process indicates a lack of an object, a theorem, that the prover is ultimately trying to obtain. Deleuze and Guattari, however, would say that this "lack" does not characterize proof at its ontological genesis. Rather, proof, at its genesis, is an unstable intensity, rather than an identity, with the potential to actualize a theorem or mathematical fact. This intensity, or affect, is proof "in itself" without a need for a theorem. In talking about a proof's "existence" beyond its signifying dimension characterized by a lack, I am referring to proof *as* this sort of intensity, one that is *asignifying*, that is, without resemblance or conformity to our more traditional ways of thinking about proof (in connection to facts and theorems or a signifying desire due to a lack of such objects). The construction of an assemblage facilitates the desiring of proof but not for facts and theorems. Rather, the desire is for proof in itself, independent from such signifying relations.

Proof has its material genesis, not in the stable, signifying world of logic and reasoning (though these do not disappear either) but, rather, in the world itself *in which the thinking mind is*

a part of but no longer privileged. This idea that thought resides outside of the human relates closely to Deleuze's notion of immanence. Concerning this idea, Philp Goodchild says "It's not the case that thought is within us, but thought is a kind of environment that we enter into and are already within" (Timeline Theological Videos, 2014, 9:09). This is how I see the genesis of proof in this dissertation. Proof is not a construction of the mind but an assemblage that begins outside of the mind in the asignifying world of intensity and affect. Roth and Maheux (2015) write, in reference to Châtelet (2010), that "in mind, nature (Being) is invisible; but nature is visible mind (beings). The linguistic field, structured according to the possibilities of the visible, therefore fails to bring us closer to the invisible and virtual Being" (p. 232). This notion of the invisible models the virtual side of proof that I explore in this dissertation. Proof at its genesis in immanence is unrecognizable to the human and does not resemble proof as we typically understand it. It is in this sense that proof, at its genesis in immanence, is invisible to us but nonetheless plays an affective role *in the becoming of a visible (or signifying) proof*.

An important part of Deleuze and Guattari's (I will use the abbreviation D&G throughout this dissertation) writing is the notion of the *body without organs* or *BwO*. Deleuze and Guattari (1980/1987), in their book *A Thousand Plateaus*, describe the BwO as an egg that contains an organism prior to its physical development. The organism prior to this organization is ontologically a very different being than the more developed organism that has more fully-formed, organized anatomy. The BwO, then, is an unorganized body that is constituted by intensities, affects, and possibilities. It has the capability to subvert, change, disrupt, and become. Deleuze and Guattari (1980/1987) often cite artists and poets, such as Antonin Artaud, in discussing issues that relate to the BwO. Art often subverts expectations and serves a liberating function that make for easy connections to the BwO (I discuss these connections in the second

chapter of this dissertation). However, for D&G, everything has a BwO. In the case of an animal such as a bird, the BwO is the organism within the egg prior to the physical formation of the wings, feathers, etc. However, there is also a sense that the BwO exists alongside the organized and the fully-formed body. That is, the BwO doesn't just precede the formed body but exists parallel to organized forms. For example, the human body may appear to be organized but it too can deterritorialize through amazing athletic feats and medical procedures. It can engage in and become what was thought to be impossible. Deleuze and Guattari (1980/1987) also warn that "You have to keep enough of the organism for it to reform each dawn…You don't reach the BwO, and its plane of consistency, by wildly destratifying" (p. 160). The BwO is a wonderous mechanism, but it can also be a dangerous one. If the body becomes too disorganized or deterritorialized, it will die. Therefore, the construction of a BwO must be done with care. In this dissertation, I view the BwO as a kind of methodology for reaching the plane of immanence and as the unstable foundation for, as Smith (2018) writes, "the production of an event" (p. 109). In the case of this dissertation, I am concerned about proof events. Therefore, I use the idea of the BwO and attempt to map it to that of proof and the proof process. Through seeing proof as a BwO and opening up what that means, mathematicians and students can find ways to reach the plane of immanence where mathematics becomes pure affect and intensity. It becomes a genesis from which the mathematically new may be fostered. This genesis does not resemble traditional axiomatic proof and yet it constitutes a new way of knowing, one that emphasizes the asignifying body, becoming, intensity, affect, and feeling rather than logical reasoning, mathematical meaning, and cognition. These aspects do not always signify anything mathematical, and yet they are important to the becoming of the mathematician and to the formation of axiomatic proof.

Why is this important? Why focus on the side of thought that does not clearly signify anything mathematical? First, this may be important for those students who may not identify or conform to more traditional modes of proof instruction. The more we can understand how mathematics partakes in the asignifying, the more we will have to help all students feel personally connected to proof and the proof process. Secondly, I argue that, in order to become one who proves, it is *necessary* to reach the plane of immanence and to think in ways consistent with the BwO. One does not find gratification in the proof process simply because one is "good" at problem-solving and obtaining "answers." *Proof is desire without relation to such objects*. There is a mysterious affect and intensity that constitutes proof as desire and as a genesis for an actual, axiomatic proof. This dissertation attempts to analyze this affect, to understand its function, and to begin to realize how this analysis is relevant to the teaching and learning of proof.

As discussed above, I utilize D&G's ideas as a kind of theoretical framework for understanding mathematical proof. I also utilize Jackson and Mazzei's (2012) book *Thinking with Theory in Qualitative Research: Viewing Data Across Multiple Perspectives* as a way to better articulate my methodological approach. This text demonstrates how a variety of theorists and philosophers, including Deleuze, can be used to make sense of qualitative data. Here, the ideas of D&G become "schematic cues" for viewing data. For example, in their chapter on Deleuze, Jackson and Mazzei (2012) use Deleuze's notion of desire as a way to understand a university administrator's silence as a way to maintain power in a conversation with a Black woman faculty member. Here, desire is not seen in a traditional Freudian sense, as a longing for something that is not yet present. Rather, desire is seen as a kind of schizophrenic proliferation of affect with the power to direct intensity and becoming to produce the new. I approach the

analysis of data in a similar way by using concepts from D&G as a frame for making sense of story data about proof.

Summary of Manuscripts

In the first piece of this dissertation (Chapter 2), I begin by unpacking some of the important ideas and concepts in Deleuze's and Guatari's work. Notions related to the body, the BwO, intensity, affect, assemblage, and sense are explored and examples are considered in areas such as music and video game software. Sense corresponds, roughly, to the meanings of statements *and* states of affairs we encounter in the world. That is, for Deleuze (1969/1990), sense and meaning involves a complicated entanglement between "*the expressed of the proposition, and the attribute of the state of affairs*" (p. 22). If language failed to express a meaning or state of affairs (the actual events that occur in the world independent of language) failed to signify any kind of quality, then there would be no sense and thought would cease (though as we will see this is not really the case). In mathematics, then, I suggest that a proof argument also has a sense or expression. Additionally, a mathematical fact or theorem (a state of affairs) also signifies its proof. That is, it has an attribute (or attributes) that points to its argument. For an argument to justify a mathematical statement, the expressions of the proof argument must correspond or adequate in some way to the attribute of the statement that is to be proved. If this correspondence or adequation fails, then the proof fails. However, it is also in this very failure that an entirely new world opens up: the world of the body, intensity, and asignifying desire. Utilizing D&G's ideas, I suggest it is this world that constitutes proof at its genesis and, therefore, we must take it into account when understanding and conceptualizing proof itself. I see this second chapter as a broad exploration and introduction to many of the concepts that will be used more explicitly in the analysis of mathematicians' proof activity.

In the second manuscript (Chapter 3), I continue the exploration I began in the first manuscript by demonstrating this theory in the context of a story written by a professional mathematician. After reviewing the theoretical ideas from the first paper, I offer analyses of excerpts from a story written by Neel (2019) and show how these excerpts demonstrate the connection between the signifying world of proof sense and the world of the asignfying intensive production that is itself a part of what proof is. This asignfying side of proof not only leads to the completion of a mathematical argument but it also leads to something entirely different: *an affective knowledge of the world that is connected to the proof process*. This knowledge or desire keeps the mathematician doing what they do (that is proving mathematical results) and opens up a more ethical approach to the study of mathematics itself.

Finally, the third manuscript situates the theoretical ideas from the previous two manuscripts in the context of the teaching and learning of proof. The main idea is to elucidate possibilities, or "lines of flight" as Deleuze and Guattari (1980/1987) might put it, regarding how the learning of proof interacts with its asignifying genesis. For example, how might bringing in ideas from philosophy and ethics aid in students' understanding of mathematical epistemology and ontology so that they have a more developed and flexible framework for approaching the proof process? How might foregrounding problem-mapping, as opposed to problem-solving, contribute to student's ability to see proof as "an opening up of possibilities" (Rota, 1997, p. 191) as opposed to simply a path to a solution. By elaborating on these potential lines of flight and their connection to D&G, I hope to begin to articulate an approach to the teaching and learning of proof that is both ethical and something that all students can feel connected to.

The primary idea that connects all three papers is the notion of immanence. Whether it is in my theorizing about proof, my analysis of story material, or in my discussions regarding the

possibilities for the teaching and learning of proof, the idea that proof and mathematics is not grounded in the transcendent world of ideal, abstract notions but rather in the world that we have access to here and now is of primary concern. This move from the transcendent to the immanent changes the way we view proof from something that is connected to correctness, mathematical duty, and meaning to an ethical dimension that foregrounds intensity, wonder, feeling, responsibility to others, and to fostering an approach to proof that has the potential to connect with those who may not signify within the context of Western notions of truth, purity, and the absolute.

Virtual Proof, Machinic Assemblage, and the Genesis of Expert Mathematical Practice

In mathematics, proof is seen as a way to guarantee or verify that a mathematical theorem is true. It also serves other functions such as explaining *why* a theorem is true as well as facilitating a manner in which mathematical meaning is communicated (DeVilliers, 1990). There is a kind of stability associated with viewing proof this way in that the transcendent, meaning-making subject is tied to an objective, logically-signifying dimension that leads to a totalization of mathematics where facts and the meanings behind these facts are foregrounded. However, such views do not open up the genetic aspects of proof that precedes (or occurs simultaneously with) its ideation. That is, proof is often seen as a given whether it is merely “a sequence of steps which leads to the desired conclusion” (Rota, 1997, p. 183), CadwalladerOlsker (2011) calls it the “*formal* meaning of proof” (p. 34), a conviction or certainty (Harel and Sowder, 1998), or “as a socially embedded activity” (Stylianides, Stylianides, & Weber, 2017, p. 247). These perspectives, however, do not take into account the genesis of proof or how these presumed conceptualizations emerge in the first place. In this piece, I apply the methodology used by Gilles Deleuze and Félix Guattari (1980/1987) from their text *A Thousand Plateaus* with regards to their emphasis on function over meaning: “We will never ask what a book means, as signified or signifier; we will not look for anything to understand in it” (p. 4). Rather, Deleuze and Guattari (1980/1987) state, “we will ask what it functions with” (p. 4). This is the goal of this piece as well, though the focus will be on proof rather than a book.

By foregrounding the discussion around functioning as opposed to meaning or identity, I aim to understand how proof operates as a kind of machine or assemblage that is capable of being disassembled, reassembled, connected with other machines, and capable of producing something new as a result of these various couplings. I do this by bringing in examples from

various, seemingly unrelated fields pertaining to video game software and music and "plugging" these machines into the machine of mathematical proof while theorizing about the resulting production from these connections. This will allow us to see how proof functions not just in relation to the production of theorems or as a way to better understand the mathematical meanings that underlie them, but as an intensity that generates the possibility for new understandings related to mathematics and beyond. I argue that it is intensity and affect, existing parallel with meaning, that best describes the genesis of proof and is therefore essential to expert (and student) mathematical practice. This type of research approach is a response to mathematical platonism and its many forms, all of which seek to "force" proof into an imitative form that merely reflects some objective structure or fact. This second chapter is designed to serve as an introduction to some of the important concepts and ideas related to the work of Deleuze and Guattari while making initial connections with mathematics and proof as well as more artistic mediums such as music and video games. As indicated previously, the point here is to show how Deleuze and Guattari's (or D&G) ideas function with proof but also how mathematics shares in the same sorts of intensities that underlie other, seemingly unrelated, pursuits. In this sense, I am constructing a sort of proof assemblage constituted by a variety of elements but with the capability to actualize new ways of thinking about mathematics.

This chapter foregrounds a fairly small group of these concepts and opening up these notions will prepare the reader for this dissertation's subsequent chapters pertaining to mathematicians' experiences with the genesis of proof as well as instructional possibilities. For example, in this chapter, I discuss the notion of sense and its role in proof as a signifying phenomenon which points to the transcendent facts and theorems that constitute mathematics itself. At the same time, I also describe how sense has the potential to collapse and lead into the

asignifying world of the body without organs and concepts related to intensity and desire. In the next section, I begin with a discussion of the platonist perspective as well as the trends in the mathematics, mathematics education, and philosophy literature supporting this worldview such as cognitive and mental processes, the subject, and axiomatic thought which expresses a more transcendent view of mathematics. In turn, I describe how the thinking of Deleuze and Guattari (or D&G) provides a more realistic, *machinic* view of proof that precedes or acts simultaneously with these transcendent perspectives. I then discuss Deleuze's (1969/1990) notion of sense in relation to proof and the manner in which sense allows for axiomatic proof to be possible followed by a discussion regarding the capability for sense to collapse into the world of the machinic, intensive body. It is this world that constitutes proof at its genesis and should, therefore, be given the same sort of attention as the study of axiomatic proof.

Deleuze and Guattari and the Reversal of Platonism

In the work of Deleuze (1969/1990), as well as Deleuze and Guattari (1980/1987), there is an attempt to undermine, or reverse, the philosophical tradition of platonism which, according to Smith, Protevi, and Voss (2023), is "an essential model of identity" (3.1 *Difference and Repetition* section). Here, there is an emphasis on ideal forms in which objects in "reality" imitate, mimic, or approximate. Chairs can vary in the material they are made of, in their size and color, etc. but all approximate the ideal form of "chair" which only exists in the transcendent. Concerning mathematical proof, there are also ideal facts (theorems) that take on a transcendent status and are approximated in our pursuit of mathematical justification. Elaborating, Rotman (2000) states:

> What manner of conviction and persuasion is there that will connect the Platonist mathematician to this ideal and inaccessible realm of objects? Plato's answer–that the

world of human knowers is a shadow of the eternal ideal world of pure form, so that by examining how what can be perceived partakes of and mimics the ideal, one arrives at knowledge of the eternal–succeeds only in recycling the question through the metaphysical obscurities of how concrete and palpable particulars are supposed to partake of and be shadows of abstract universals. (p. 31–32)

The work of the mathematical prover is the engagement of these "shadows" via the inferential process of deductive argumentation, the axiomatic method. The axioms, lemmas, propositions, and theorems that become codified are the ideal forms that are approximated when mathematicians and students engage in this method. For D&G, what this means is that proof as represented in the axiomatic approach is founded on a presupposed identity, an assumption that these objects somehow already exist as stable entities to be discovered or, in the ideal space of the mind, invented via cognitive construction. This presupposition, however, fails to engage in its own genesis and to make sense of how these identities came to be in the first place. For platonism, the formulation begins with assumed axioms as a stable foundation of which to then generate complexity and nuance through the deduction of lemmas and theorems (which themselves become stable identities of which further results can be generated). Variations of this approach exist such as Hegel's dialectics which posits that the generation of the new results from two opposing, yet stable, ideas, thesis and antithesis, entering into a synthesis. Sameness first, then difference. However, for Deleuze (1969/1990), this is reversed so that complexity, the unstable, and the nuanced is made primary followed by the stratification of stable identities. That is, D&G enact a *reversal* of the platonist formula: *difference first, then sameness*.

What are the consequences of this move? By taking the view that the genesis of things begins in difference rather than stable identity, it becomes more difficult to accept overly-rigid

worldviews or metanarratives. An important aspect of post-structural philosophy (of which D&G are a part) is the pursuit of deconstruction which seeks to dismantle such metanarratives that impose stable and fixed meanings (some of which are oppressive, colonizing, and reductive) on the world and on others. By reversing platonism and assuming the complexity of difference first, we reduce our commitment and adherence to dogmatic worldviews and axiomatic thought that keep us within these artificial boundaries.

Another aspect of D&G's, reversal of platonism is that it opens up a tendency to make primary materiality over language and linguistics which are potential avenues of the platonic idealism that D&G are trying to counter. Linguistics and semiotics, particularly the structuralist brand as established by thinkers such as Saussure and Frege, tend to essentialize the relationship between signifier and signified (that is, the reference and the referent or the word and the actual thing that the word points to) since words "derive meaning from their relationship to other words in the system...a negative relationship, where signs get their meaning from *what they are not*" (Jackson & Mazzei, 2012, p. 69). This sets up a dynamic that is similar to Hegel's dialectics concerning the synthesis of opposing forces. As already mentioned, this type of approach assumes, in advance, the existence of the stable prior to their synthesis or difference. Frege also supports this essentialization by dealing in sense as "the condition of the true" (Smith, 2006a, p. 138) and as "a matter of adequation or reference" (Smith, 2006a, p. 139). For Frege, truth is the assumed stable identity or essence. However, as stated by Smith (2006a) regarding Deleuze's (1969/1990) "difference first" approach:

> Rather than utilizing a method of conditioning, which would presume truth as a "fact" and then seek its conditions, Deleuze holds that philosophy must adopt a method of

> genesis: truth must be seen to be a matter of production within sense (method of genesis) rather than adequation to a state of affairs (method of conditioning). (p. 139)

Note that Frege's "method of conditioning" involves adequation, a type of mimicking in a similar vein to how "reality" approximates the world of the "forms" in Plato's view or, in the case of mathematical proof, the adequation of a proof construction to a particular mathematical fact or theorem. However, D&G's philosophy does not involve mere adequation. Rather, it involves the *production* of truth in genesis, that is, in difference, Truth is not merely taken for granted but is preceded by a process of becoming. Further elaboration on D&G's genetic theory will be discussed in subsequent sections.

In the mathematics education and philosophy literature, mathematical proof is often described in terms of meaning, essences, and the transcendent. For example, Rota (1997) states that "Everybody knows what a mathematical proof is. A proof of a mathematical theorem is a sequence of steps which leads to the desired conclusion" (p. 183). Nathanson (2009) provides a similar statement about proof in that it "typically consists of a series of assertions, each leading more or less to the next, and concluding in the statement of the theorem" (p. 8). Following Smith's (2006b) discussion of theorematics, such conceptualizations are "defined statically, in Platonic fashion, in terms of its essence and its derived properties" (p. 148). Other transcendent views of proof also exist. For example, Brown (1997) emphasizes the use of pictures as a way in which one may justify a statement, indicating that "*some 'pictures' are not really pictures, but rather are windows to Plato's heaven.…*As telescopes help the unaided eye, so some diagrams are instruments (rather than representations) which help the unaided mind's eye" (p. 174). Under these notions, a proof's objectivity and its respect for logical rules and deductions are central.

While an objectivist standpoint should seem familiar to those in the mathematical community, there has been an increased interest in seeing proof as also connected to the individual. For example, in CadwalladerOlsker (2011), the author explores the subjective side of proof citing works such as Raman (2003) and Harel and Sowder (1998) in which conviction and proof as mathematical meaning appear to be foregrounded. While there are differences in emphases on the subjective and the objective across these works, it is likely that many of these researchers focusing on the study of proof see both sides of this duality as important and relevant. For instance, Czocher and Weber (2020) propose an interesting cluster category approach to conceptualizing proof where various objective aspects (e.g., proof is an a priori justification) and subjective aspects (e.g., proof is a convincing justification) are entangled with each other in a dynamic manner. Kidron's and Dreyfus' (2014) *proof image* involves both sides of the subject/object binary in which "the interplay between the learner's intuitive and logical thinking as well as the construction of knowledge that results from and enables progress of this interplay" (p. 299) is centered. These perspectives of proof, while having some emphasis on meaning-making and cognitive construction, also reflect platonism by placing the world of the forms within a transcendent subject (the mathematician). The work of the mathematician, then, is to imitate an ideal mental construction via the written proof. As Rotman (2000) puts it:

> conviction and persuasion appear as the possibility of a replay, a purely mental reenactment within this one subjectivity: perform this construction in the inner intuition of time you share with me, and you will–you must–experience what I claim to experience. (p. 28)

Platonism, therefore, plays a role in many approaches to the philosophical understanding of proof. It is an imitative form that is discovered as a property of a reality, whether this reality is

located in an ideal space external to the individual mind or within the interiority of the mathematician. If we follow Deleuze's (1969/1990) approach of reversing this platonism as it pertains to mathematical proof, what we do not have is a stable, presupposed imitative form but, rather, a *dynamic genesis* of difference. In the next section, I provide an overview of Deleuze and Guattari's ideas about machine, body, and intensity. These notions will guide our later conversations regarding how the genesis of proof is constituted.

Proof as Machinic Assemblage, Body, and Intensity

In *Anti-Oedipus* and *A Thousand Plateaus*, D&G utilize the notion of machine as a way to understand what bodies are capable of achieving. Concerning this, Smith (2018) writes:

> Whereas Cartesian mechanism has tended to emphasize the formal and substantial aspects of machines, the actual properties they have, and the predictable movements that this entails, Deleuze and Guattari's "machinism" emphasizes instead the "virtual" side, the capacities that machines have to do something *other* than what they were designed to do. (p. 100).

Regarding this aspect of the virtual machine, D&G take inspiration from Spinoza's *Ethics* regarding the possibilities of the body. While it may seem that bodies have many physical limitations, there are times when they are capable of performing acts that were not thought possible. Therefore, for both Spinoza and D&G, there is a latent, virtual potential in every body to exist and live in ways that were not thought possible or, perhaps, not originally intended. The body without this virtual potential is ruled by organization. Deleuze and Guattari (1980/1987) writes "The *judgment of God*, the system of the judgment of God, the theological system, is precisely the operation of He who makes an organism, an organization of organs called the organism" (p. 158). A body ruled by the organization of the organs is an organism, but the body

ruled by the virtual possibilities of the machine is "the *becoming-machine of the organism"* (Smith, 2018, p. 109). Smith (2018) associates this latter body with D&G's notion of the Body without Organs (BwO), an entity that is in a constant state of subversion, movement, and deterritorialization. For example, in *The Logic of Sense*, Deleuze (1969/1990) discusses a "*language without articulation*" (p. 89) that foregrounds the schizophrenic sounds of the body over meaning and sense. The schizophrenic is closer to that of the BwO in that the body of the schizophrenic is not ruled by the identity of organization, that is, a body imbued with meaning. Rather, the schizophrenic body cannot distinguish meaning and sense from the body since it has been deterritorialized. That is, it has resisted organization and the subjectifying forces that seek to reterritorialize and place it under the control of the organs, society, capitalism, and other norms.

To be deterritorialized, however, is not always something associated with the positive. Deleuze and Guattari (1980/1987) indicate that one may deterritorialize so much that there is nothing left: an *empty*, *catatonic body*. For example, students of mathematics may become so discouraged with the material they are learning that they no longer really engage with it, though perhaps just enough to receive passing grades in their classes. Even experts can have these experiences, though they are perhaps usually brief and temporary situations. Consider a nonmathematical example: individuals who become so attached to a particular worldview that they have a difficult time reterritorializing elsewhere and life ceases to be dynamic. Deleuze and Guattari (1980/1987) write:

> You have to keep enough of the organism for it to reform each dawn; and you have to keep small supplies of significance and subjectification, if only to turn them against their own systems when the circumstances demand it, when things, persons, even situations,

> force you to; and you have to keep small rations of subjectivity in sufficient quantity to enable you to respond to the dominant reality. Mimic the strata. You don't reach the BwO, and its plane of consistency, by wildly destratifying. That is why we encountered the paradox of those emptied and dreary bodies at the very beginning: *they had emptied themselves of their organs* instead of looking for the point at which they could patiently and momentarily dismantle the organization of the organs we call the organism. (pp. 160–161)

The body as machine, or the body *in relation* to the BwO, is most healthy when, even in its extreme, deterritorilized state, it retains a potential to reform and territorialize elsewhere, to plug into other machines. How might mathematical proof behave like a machine or assemblage rather than as an organized body that merely signifies a particular mathematical fact? Rota (1997) seems to suggest that proof emerges not from processing a definitive, formal argument but rather "after digging deep and focusing upon the possibilities of the theorem." (p. 191). Rota (1997) even offers examples such as Desargue's Theorem:

> After an argument that runs well over one hundred pages, Baker shows that beneath the statement of Desargues's theorem another far more interesting geometric structure lies concealed. This structure is nowadays called the Desargues configuration. An explanation of the Desargues configuration in terms of theorems and proofs is lengthy and unsatisfactory. The Desargues configuration is better understood by meditating upon a figure displaying incident straight lines in the plane, more than 50 straight lines if I remember correctly. Once the Ideenkreis of the Desargues configuration is intuitively grasped, one understands the reasons that lay concealed beneath the statement of Desargues's theorem. (pp. 191–192)

Eventually, Rota (1997) states:

> Thus, even as simple a formal statement as Desargues's theorem is not quite what it purports to be. The statement of Desargues's theorem pretends to be definitive, but in reality, it is only the tip of an iceberg of connections with other facts of mathematics. The value of Desargues's theorem, and the reason why the statement of this theorem has survived through the centuries, while other equally striking geometrical theorems have been forgotten, lies in the realization that Desargues's theorem opened a horizon of possibilities that relate geometry and algebra in unexpected ways. (p. 192)

Not only is Desargue's configuration key for Desargue's Theorem but it also taps into the machinic dimension of the theorem, allowing for a "horizon of possibilities" across mathematical fields. Like a BwO, Desargue's Theorem can be disassembled and reassembled in order to reterritorialize or plug itself into other spaces that generate further mathematics. Rota (1997) ends his point with a profound insight about formal mathematics: "what an axiomatic presentation of a piece of mathematics conceals is at least as relevant to the understanding of mathematics as what an axiomatic presentation pretends to state" (p. 192). From this point, Rota (1997) concludes that there is no such thing as "definitive proofs." This is because the virtual side of a proof, which exists in parallel with the supposed "definitive" proof itself, is never fully stable or in balance. There is always the possibility that a more desirable proof configuration will be discovered, which may then open new directions in mathematics.

In the passage above, Rota (1997) deterritorilizes the proof of Desargues's theorem, eschewing the organization typically associated with this type of scientific work, the correlation of a proof with a particular mathematical fact, and, instead, couples the proof-machine for this particular theorem with different areas of mathematics in "unexpected ways" via the Desargue's

configuration. Rota (1997) sees proof as something that is in relation to the BwO and to elucidate this relationship, more focus needs to be placed on the possibility for proof to exist in the virtual. Rota (1997) elaborates further, stating, concerning Fermat's last theorem, that

> The value of Wiles's proof lies not in what it proves, but in what it opens up, in what it makes possible.
>
> Every mathematician silently knows that such an opening up of possibilities is the real value of the proof of Fermat's conjecture. Every mathematician knows that the computer verification of the four color conjecture is of considerably lesser value than Wiles's proof, because it fails to open up any significant mathematical possibilities. But most mathematicians will pretend that the value of a proof, as well as its future possibilities, are non-mathematical terms devoid of formal meaning, and will thus be reluctant to engage in a rigorous discussion of the roles of value and possibility in a realistic description of mathematics.
>
> We propose instead that a rigorous version of the notion of possibility be added to the formal baggage of metamathematics. One cannot pretend to disregard possibility, by arguing that the possibilities of a mathematical result lie concealed beneath formal statements. Nor can one dismiss the notion of possibility on the ground that such a notion lies beyond the reaches of present day logic. The laws of logic are not sculpted in stone, eternal and immutable. (p. 191)

It is the goal of this paper to further investigate proof as an "opening up of possibility." However, in addition to seeing the possibilities for proof to engage in machinic functioning with other mathematical fields, I also see proof as a machine with the potential to plug into other machines that, at first glance, seem fairly unrelated to proof and mathematics. I demonstrate this

by exploring areas related to video games and music. That is, I plug these machines into the machines of mathematical proof, to follow the methodology described in Deleuze and Guattari (1980/1987) concerning function over meaning and to see what is produced from such couplings. That is, to see what proof is capable of. We find that proof, when connected with other machines, produces, at the level of dynamic genesis, intensities which “cannot be achieved through the ordinary exercise of our sensibility. Intensity can be remembered, imagined, thought and said. Intensities are not entities, they are virtual yet real events whose mode of existence is to actualize themselves in states of affairs” (Boundas, 2022, pp. 133-134). Similarly, referencing Deleuze (1968), Hughes (2008) states that “Intensity clearly has the directive role” (p. 123). In this paper, I show how the “real value” of proof is in its ability to transmit intensities and that this transmission is at the very genesis of mathematics. In the next section, I discuss Deleuze’s (1969/1990) notion of sense and event which will prepare us for a subsequent discussion of the musical project Boards of Canada (BoC) and how this music models the sorts of intensities produced at the virtual level of proof.

Series, Sense, and Event-Effects

In *The Logic of Sense*, Deleuze (1969/1990) unpacks the notion of series and their interactions. While the term seems to suggest a kind of gesture or sequence of some kind, anything can be modeled with series. An individual is always in a constant state of producing themselves in time and space, moving toward something. Becoming, just as a mathematical sequence or function tends toward some value. The key here is the “tending toward” as opposed to the value itself that is approached. For example, a book tends toward certain meanings. A body, comprised of various biological processes and flows, tends toward certain states (e.g., illness, recovery from disease, various developmental milestones). That is, a body can be

conceptualized as a movement, even if it is physically at rest. A proof involves moving in and out of stages and mathematical detours. Poxon and Stivale (2011) indicate that "The series is important to Deleuze because it instantiates a mode of organization of difference that avoids the pitfalls of representation, within which difference is tamed by the mechanisms of resemblance, identity, analogy and opposition" (p. 70). Deleuze's notion of series is one way that he foregrounds difference even in identities that appear to be static at first glance. It is in this sense that everything can be thought of in terms of series.

Series do not merely exist. A substantial portion of *The Logic of Sense* is examining how these series interact and produce something new. For Deleuze (1969/1990), sense is produced when two or more series come together or collide with one another. In *The Logic of Sense*, Deleuze (1969/1990) defines sense as "*both the expressible or the expressed of the proposition, and the attribute of the state of affairs*" (p. 22). The word "expressed" seems to be used in a manner similar to that of "meaning" while "the attribute" is associated with "the quality which the proposition denotes" (p. 22). That is, attributes are the actual qualities that exist outside of language and the mind. It's important to note the word "both" in Deleuze's definition. Sense is always *the boundary* between the expressed of the proposition and the attribute of the state of affairs. It is where these two series meet. Sense is like the surface of the ocean; it has zero dimension while also constituting its own frontier. However, as Brady (2017) notes "sense is all that is holding words and objects apart, *and* is all that is holding them together" (Reversing Sense section). Therefore, sense has both a stabilizing as well as a tenuous relationship between states of affairs and propositions. Smith, Protevi, and Voss (2023) offer an elaboration on an example of sense from Deleuze (1969/1990):

> I can attribute the proper name "Battle of Waterloo" to a particular state of affairs, but the battle itself is an incorporeal event (or sense) with no other reality than that of the expression of my proposition; what we find in the state of affairs are bodies mixing with one another—spears stabbing flesh, bullets flying through the air, cannons firing, bodies being ripped apart—and the battle itself is the *effect* or the *result* of this intermingling of bodies. (3.2 *Logic of Sense* section)

Regarding the term "effect," Deleuze (1969/1990) notes that

> *effects* are not bodies, but, properly speaking, 'incorporeal' entities. They are not physical qualities and properties, but rather logical or dialectical attributes. They are not things or facts, but events. We can not say that they exist, but rather that they subsist or inhere (having this minimum of being which is appropriate to that which is not a thing, a nonexisting entity). (pp. 4-5)

From this description effects and events appear to have a metaphysical status. They do not exist yet they have being. To return to Smith's, Protevi's, and Voss' (2023) discussion, the event of the battle cannot be found in its actual states of affairs or within the propositions that refer to it. Rather, the battle is found in the effect that the intermingling bodies produce. Observe that the description of "bodies mixing with one another" discussed in Smith, Protevi, and Voss (2023) may also similarly describe other events such as a football game. However, the meaning of the proposition "a football game" and "The Battle of Waterloo" are quite different. Therefore, because the sense of a state of affairs must also involve the meaning of its proposition, a football game and The Battle of Waterloo are not the same from the perspective of the event-effect. How do these concepts, that of series, sense, and event-effects, relate to mathematical proof? I explore some possible examples in the next section.

Sense as Signification and the Interaction Between Theorem and Proof

Brady (2017) offers an interesting example of sense involving a shopping list. Suppose I were to find a notepad with a list (or series) of the following items: "milk," "batteries," "eggs," and "tomatoes." Many individuals would infer that the list (or series) signifies a shopping list as opposed to, say, a poem. To take the list as a poem would be to take it "*in the wrong sense*" or meaning (Shopping Lists section). To understand this list of words as a shopping list (and not as a poem), I know, then, to associate these words with the actual objects found in the grocery store rather than in a figurative or symbolic sense. If I reverse the direction and then look at the actual groceries themselves (rather than the list of words that denote them) as the signifying series, what do these actual objects signify? This time, Brady (2017) indicates, it likely won't be a grocery list but, instead, "it would be a tomato and milk protein shake with crushed battery acid: *energy drink of kings*" (Reversing Sense section). This strange drink is akin to the "snark" in Alice's Adventures in Wonderland. Things can signify paradoxical elements that do not belong in language but still have a sense to them.

Let's try to construct a similar example to that of the shopping list, except one that is relevant to mathematical proof. Consider a mathematical statement *an even integer plus an even integer is equal to an even integer* and its proof. The proof, that is the signifying series, consists of lines of inferences that signify (that is prove) the statement (the series of signifieds). The proof may be given in the following manner:

- Let x and y be two arbitrary even integers (where "even integer" is defined as an integer that is a multiple of 2).
- So, by definition of even integer, $x = 2n$ and $y = 2m$ where n and m are arbitrary integers.
- Therefore, $x + y = 2n + 2m = 2(n + m)$

- Since $x + y = 2(n + m)$ is also, by definition, an even integer, I am done.

The above steps enter into differential relation with each other and signify the mathematical statement to be proved (remember that this statement constitutes the signified series). How do the lines above do this? Via ideas concerning *common factors*: If two numbers share a common factor, their sum will also have that factor and since, in this case, the common factor is 2, the sum will also have a factor of 2 and is therefore even. The real meaning of the proof is not in the statements above but in the sense or expression that it communicates implicitly. If I did not understand the role of common factor in the argument, then I may not really understand the sense or meaning of the proof, and the passage to the mathematical result (the proved statement) would not occur. As long as I understand the role of common factors and its sense as the expressed of the proposition (the proposition being the signifying series or lines that constitute the given proof argument), I am allowed successful passage to the signified statement. It's also important to observe that the signifying series (in this case the proof steps) is always in excess of the signified series (the statement to be proved). As Deleuze (1969/1990) notes, there is a kind of reduction or lack that results in passing from the signifying series to the signified series.

What happens if I were to reverse the order? This is important since Deleuze (1969/1990) makes clear that sense includes both the *expressed of the proposition* (what I just analyzed in the previous paragraph) and *the attribute of the state of affairs*. Therefore, part of the sense of proof is examining it from the side of the statement to be proved and toward its justification. So, let's assume I am only given the actual mathematical statement *an even integer plus an even integer is equal to an even integer* (but not the proof) and I want to know what it signifies (in this case, what the proof of the statement is). The statement, which now acts as the signified series due to the reversal, also bears sense in that the left side of the equality (the sum of any two even

integers) enters into relation with the right side of the equality (also an even integer). From this relation, a sense or attribute of this given fact or state of affairs is produced that can hopefully lead to the formation of a mathematical proof. However, the question is this: Is it the same sense that is produced from the proof steps as discussed previously? Maybe but I have less information to work with in this case. Perhaps, given the statement (but again not the proof of the statement) our minds would wander to the idea of *pairing objects*. Since an even number of objects are constituted by pairs with no remainder and the sum of two groups, both consisting of an even number of objects, is also constituted by pairs, I can try to construct a more rigorous proof based on this sense. However, the sense of object pairing does not seem to be exactly the same as the sense associated with common factors as the former involves a more visual understanding while the latter involves algebraic manipulation. Therefore, it seems very possible that I can get at least slightly varying meanings depending on the direction of signification (from proof to statement or from statement to proof). Since sense takes into account both of these directions at once, the result is a degree of imbalance and potential contradiction whenever I consider proof as sense or meaning (as opposed to simply a sequence of "correct" inferences).

This idea of imbalance seems to correspond to Rota's (1997) view concerning the following question: Which is more important, theorems or their proofs? By analyzing certain areas of mathematics, some (e.g., algebra, number theory) hold to more formality than others (e.g., geometry), Rota (1997) concludes that one is not primary over the other but, rather, "theorem and proof play the role of Tweedledum and Tweedledee. In this sense we may assert that theorem and proof are exchangeable" (p. 191). There are two important points to summarize here:

1) Our understanding of an already given proof *emerges from* the sense that forms when the proof steps enter into relation with one another (the meaning or an expressed of the proposition) which then allows passage to the result (the statement or mathematical fact or state of affairs that is obtained). On the other hand, our construction of a proof, for a given mathematical statement, *emerges from* the components of the theorem statement (the attribute of the theorem as a state of affairs) entering into relation with one another, eventually allowing passage to a fully developed proof argument that justifies the given statement. Following Smith (2022), this formulation seems to correspond with Deleuze's (1969/1990) notion of *static genesis*. That is, the sense of a proof is the result of propositions or states of affairs.

2) Therefore, following Deleuze's (1969/1990) idea that sense has two sides, proof sense involves *both* the comprehension of given proofs (from the expression of language) and the production of proofs (from the attribute of states of affairs). That is, proof sense is double-sided also. Proof sense is stable enough to allow passage from a proof to its result (and vice-versa) but also unstable in that different senses are obtained depending on whether I am on the comprehending side or the production side of the signification. Therefore, proof sense is not fully convergent.

As noted in Smith (2022), some philosophers such as Frege and Russell saw sense as "the *condition* of truth" (p. 9) rather than the *result* of propositions and states of affairs. Smith (2022) writes:

> In other words, Deleuze attempts to provide a *genetic* account of truth, rather than seeking the conditions of truth as a mere 'fact'. Put simply, truth must be seen to be a

> matter of *production* within sense (method of genesis) rather than *adequation* to states of affairs (method of conditioning). (p. 10)

In relation to proof, whether or not the sense of a proof is the *result* of propositions and states of affairs or the *condition* for such formal aspects, it still plays a signifying, adequating, or connecting role. That is, the static genesis of proof still serves to ensure a passage between proofs and theorems. It points and anchors itself to the transcendent facts of mathematics. However, I will eventually demonstrate that mathematical proof involves a *dynamic genesis* which involves how truth and proof sense are bodily productions rather than simply connections between mathematical propositions and states of affairs.

The results given above also seem to support certain findings in the study of logic. For example, Gödel's Incompleteness Theorems state, roughly, that no logic system can exist that is both *complete* and *consistent*. A complete system is always able to show the truth or falsity of a statement (though the system may not necessarily avoid contradiction), while a consistent system produces results free of contradiction (though the system may not necessarily be complete). Therefore, Gödel found that for any logic system, there is always a degree of incompleteness and/or error involved. What this means is that there will always be the potential for holes, errors, paradoxes, imbalances, and divergences in our ability to demonstrate mathematical thought. However, it seems that Deleuze would see this as a potentially good thing. If there existed a system that exhibited completeness and consistency, then there would be no mathematical meaning because there would be no collision of series. That is to say, there would be no mathematical sense. While incompleteness and inconsistency in mathematics poses challenges to mathematicians, they are necessary for mathematical meaning to be made in the first place.

To some extent, the research on proof as given in the literature resonates with the notion of proof sense as outlined above. The subjective and objective components of proof come together in this literature like the bringing together of two forces or series. Even notions such as key ideas (Raman, 2003) and representation systems (e.g., Lockwood, Caughman, and Weber, 2020) seem to suggest that proof is not something concrete but rather emerges out of an intuition that simultaneously does not guarantee correctness while also helping to produce a sense of conviction that leads to objective mathematical results. However, what is missing from the proof literature is explicit theorizing about proof sense and its genesis as opposed to presupposing its existence through the definition of various, often cognitively-related constructions. Additionally, the results described above suggest that proof has the potential to become something beyond the mere statement and processing of facts and should take into account the destabilizing potential that proof sense has to offer.

In the next section, I discuss proof in conjunction with the musical project known as Boards of Canada (BoC). I discuss this work in terms of series and event-effects as well as how the BoC community taps into the machinic nature of this music in order to discover new intensities. I then associate BoC and their community with the community of mathematicians who generate mathematics through proof. In pursuing this line of thinking, I do not simply associate mathematical proof and BoC for the purposes of clarifying analogy (imitation), but rather to plug the machine of BoC into the machine of mathematical proof in order to further understand proof's function rather than as an organized structure.

Musical Collisions: Proof and Intensity

Boards of Canada is an electronic music project consisting of Michael Sandison and Marcus Eion. O'Neal (2018) describes their music as "childhood daydreaming mixed with '70s

kitsch; a wistful nostalgia refracted and distorted; an artificial recreation of a (probably false) memory" (para. 5). While their work is playful, dreamlike, and nostalgic it also comments on contemporary issues. For example, in one particular track (see geokaker, 2019, 14:56), a news report regarding the deaths of "19 people" in Iraq is juxtaposed with what sounds like a video arcade from the 80's or 90's, perhaps suggesting society's apathy toward war in an era where media violence is a part of everyday life. The use of the collage approach also places BoC's work close to the spirit of D&G's notion of machinic assemblage. By bringing together two different media machines (that of television news and the arcade), an intensity is produced from this coupling that suggests a particular worldview. Television news and arcades function with each other to form new political ideas.

Another way that BoC pursues a compositional methodology of function is with regard to how their albums transition from one track to another. Often, a BoC album or fan mixtape does not simply "stop" between tracks. Rather, there is one or two seconds in which the end of one track merges with the next. Sometimes, this is accomplished simply through a "fade out" and "fade in" pattern. Some of these transitions are abrupt and involve a felt sense of "breaking through" or escape from one space to another. Deleuze's (1969/1990) notions of series and event-effects are useful here in that the two tracks that meet in this brief sonic intersection can be modeled as two series (the first acting as the signifying series while the second acting as the signified series). The border between the two series, where the intersection occur, is the produced sense of the collision. This moment of intersection (which usually lasts no more than a second or two) is an entire frontier that "moves in both directions at once. It always eludes the present, causing future and past, more and less, too much and not enough to coincide in the

simultaneity of a rebellious matter" (Deleuze, 1969/1990, p. 2). This collision is an event but felt as an effect which, Deleuze (1969/1990) states, is:

> sonorous, optical, or linguistic "effects"–and even less, or much more, since they are no longer corporeal entities, but rather form the entire Idea. What was eluding the Idea climbed up to the surface, that is, the incorporeal limit, and represents now all possible *ideality*, the latter being stripped of its causal and spiritual efficacy. (p. 7)

Events and their accompanying effects, then, are a kind of "breakthrough" that cannot be explained causally but rather via an eruption into and through the boundary of sense. The production of sense is not a smooth event but one that always involves a messy and sometimes explosive correspondence/divergence between series. These effects are intensive and cannot be grasped or explained in language. In the case of BoC, intensities flow throughout the assemblage of an album and are generated through the collisions that occur between tracks. In this way, a BoC album resembles that of a mathematical proof: the musical pieces (or mathematical statements) align themselves in just the right manner to produce a particular musical intensity or event (or in the case of proof, a mathematical sense or meaning) that results in an album (or theorem). Sense is relevant here in that there is a sort of connecting function that occurs between tracks, just as there is between a proof and its theorem. However, given the sonic nature of BoC's work, this sense seems to be more produced and more in line with an intensive, bodily production than purely a matter of adequation or meaning-making.

The focus here on transitions in the music of BoC (and the resulting fan contributions) foregrounds an important aspect of intensity which is its lack of extension. Working from an example from Smith, Protevi, and Voss (2023), if a container of water at 90ºF were to be evenly distributed into two smaller but equally sized containers, the volume of the water would be

divided by two. However, the temperature of the water would not be. Rather, the temperature would remain the same. Temperature is an intensive quantity because it cannot be divided in this way and retains its value along a continuous space. Unlike volume (and other sorts of measurements), temperature cannot be broken down further into smaller quantities and is therefore difficult to control and grasp. That is to say, temperature does not have extension. Boards of Canada's music could be described in this way, also. Any attempt to separate the tracks from any given album or mixtape cannot be done without fundamentally altering its sound and musical character. The manner in which tracks are entangled with one another ensures the intensive aspect of BoC's music. This is why each fan mixtape has its own unique sound and feeling even though many of its tracks remain unchanged. While it is, perhaps, easier to understand how a proof generates meaning, in what ways does the sense of a proof produce an intensity? In Raman (2003), the author discusses the notion of a proof's *key idea* which

> is an heuristic idea which one can map to a formal proof with appropriate sense of rigor. It links together the public and private domains, and in doing so gives a sense of *understanding* and *conviction*. Key ideas show *why* a particular claim is true. (p. 323)

Raman (2003) writes: "The connection is rather straightforward: the heuristic idea is essentially private, the procedural idea is essentially public, and the key idea provides the link between the two" (p. 324). Notice the role of series in this definition: there is public and private as well as understanding and conviction. Also notice how the key idea connects the two series. It is in this way that the key idea, then, is the boundary between these series. It is an event that marks the virtual structure of proof that precedes its actualized form in the axiomatic proposition. I argue that the key idea produces *understanding* on the side of the public, signified series and *conviction* on the side of the private, signifying series. That is to say, understanding is closer to that of

meaning while conviction is a more bodily sense or quality on the side of the states of affairs. Conviction is an intensity rather than simply a cognitive state. Bell (1976) seems to support this notion, stating that

> Conviction is normally reached by quite other means than that of following a logical proof.…and I would suggest that conviction arrives most frequently as the result of the mental scanning of a range of items which bear on the point in question, this resulting eventually in an integration of the ideas into a judgement. Proof is an essentially public activity which follows the reaching of conviction. (p. 24)

Bell's (1976) suggestion that proof follows conviction suggests that conviction is not the same as obtaining the meaning of a proof as a proposition. That conviction precedes proof suggests that it is still a part of sense and connected to the key idea. It is, however, the result of coming into contact with the quality of a proof as a state of affairs as opposed to a proposition. Also note that in Bell's (1976)'s discussion above, conviction is something that is "reached" through "mental scanning." This further suggests that, perhaps, conviction is not just a binary state but, rather, a process that can become "more or less." The integration of series is akin to the intensive relationship between tracks on a BoC album. Just like BoC, the key idea requires a careful integration of series. If this integration is not handled with care, then the proper affect or intensity will not be transmitted on a BoC album and the conviction and meaning won't obtain in a mathematical proof. Intensity, as it pertains to proof or music, is about a careful calibration of virtual forces on the plane of immanence. Virtual proof is where this calibration takes place and what is ultimately generative of the axiomatic proof that is given in textbooks, research papers, and the mathematics classroom. Sense is important, in both the music of BoC and in

mathematical proof, but this sense does not exist statically, it is the result of a precise production or integration that produces an album or a mathematical result.

Returning to BoC, there is one more important aspect of their music that is important: the fan community. I will discuss this aspect and then, again, plug these notions into the proof machine to further see what proof is capable of. As a simple Google or YouTube search will show, fans of Boards of Canada take great interest in developing their own mixtapes consisting of various BoC tracks across their discography. Similar to their official releases, fans seem to preoccupy themselves with the transitions between tracks, engineering these collisions in just the "right way" as to produce new intensities that were not present in official albums. That is, fans utilize a methodology that was already present in BoC's work and apply these same methods to "compose" their own transitions using preexisting material. These rearrangements of the source material create new musical intensities, some of which are very popular across the community. In some ways, this type of activity resembles Rota's (1997) sentiment that "The search that mathematicians are now beginning to perform upon the proof of Fermat's last theorem is meant to disclose 'what' it is that is 'really' being proved. This search will keep mathematicians busy for a long time" (p. 195). This search for the hidden meaning of Fermat's last theorem is much like the musical work done by BoC fan community. Fans are "searching" the music of BoC in order to discover the juxtaposition that produces just the "right" intensity. The resulting production is something that can no longer be exclusively attributable to BoC. The BoC mixtape, in the abstract rather than any particular iteration, is the plane of immanence for BoC and their community. What might this plane look like with regards to proof?

What I have done in this analysis is deterritorialize the notion of a mixtape to that of a *virtual concept* that no longer resembles what it once was (a specific arrangement of musical

pieces). Now, the mixtape is a kind of musical network or assemblage such that intensities can easily flow, transmit, produce, and be produced. By plugging the concept of the mixtape into the concept of proof, I also deterritorialize the traditional notion of proof and see it, also, as a BwO where intensities reside (though these intensities may differ in type and magnitude). The idea of proof as a BwO is a complicated one, particularly since the immanent is virtual and not directly accessible. Like effects, the plane of immanence does not "exist" but, rather, "insists" or "subsists" (Deleuze, 1990). To explore this issue of the deterritorialized proof (or virtual proof) further I turn to Deleuze's and Guattari's discussion of *problematics* as opposed to *axiomatics* and how the former allows for a more realistic understanding of mathematical proof as a becoming as opposed to just a linear, inferential reasoning process.

Problematics and Proof

In Smith (2006b), the author discusses Deleuze's notion of problematics and axiomatics:

> The fundamental difference between these two modes of formalisation can be seen in their differing methods of deduction: in axiomatics, a deduction moves from axioms to the theorems that are derived from it, whereas in problematics a deduction moves from the problem to the ideal accidents and events that condition the problem and form the cases that resolve it. (p. 145)

Regarding problematics, there seems to be a more exploratory aspect to this sort of mathematical reasoning that is more concerned with opening up the possibilities of a problem as opposed to pinning down a concrete solution. A concrete solution may be obtained by working within problematics, but the central concern is to map out a problem first. Smith (2016b) elaborates further concerning an example of problematics that Deleuze had elaborated on:

> In 1824, Abel proved the startling result that the quintic was in fact *unsolvable*, but the method he used was as important as the result: Abel recognized that there was a pattern to the solutions of the first four cases, and that it was this pattern that held the key to understanding the recalcitrance of the fifth. Abel showed that the question of 'solvability' had to be determined internally by the *intrinsic* conditions of the problem itself, which then progressively specifies its own 'fields' of solvability. (p. 160)

Problematics, it seems, emphasizes the possibilities of the problem's internal structure rather than the derivation of a result which appears to be the primary concern of axiomatics.

In considering the more traditional axiomatic approach, Thurston (1995), a mathematician, describes how a particular field that he had been pioneering developed over time and how the conventional theorem-proof formula in much of mathematics research can serve to close down certain research areas:

> First, the results I proved (as well as some important results of other people) were documented in a conventional, formidable mathematician's style. They depended heavily on readers who shared certain background and certain insights. The theory of foliations was a young, opportunistic subfield, and the background was not standardized. I did not hesitate to draw on any of the mathematics I had learned from others. The papers I wrote did not (and could not) spend much time explaining the background culture. They documented top-level reasoning and conclusions that I often had achieved after much reflection and effort.…This created a high entry barrier: I think many graduate students and mathematicians were discouraged that it was hard to learn and understand the proofs of key theorems. (p. 35)

The result, according to Thurston (1995) was that

> a dramatic evacuation of the field started to take place. I heard from a number of mathematicians that they were giving or receiving advice not to go into foliations–they were saying that Thurston was cleaning it out. People told me (not as a complaint, but as a compliment) that I was killing the field. Graduate students stopped studying foliations, and fairly soon, I turned to other interests as well. (p. 35)

It is clear that the above discussion is not merely an issue of mathematical difficulty, it is an issue of method. The axiomatic approach to proof insists on a rigorously, linear workflow that is incapable of participating in the intensive. As Thurston (1995) concludes:

> When I started working on foliations, I had the conception that what people wanted was to know the answers. I thought that what they sought was a collection of powerful proven theorems that might be applied to answer further mathematical questions. But that's only one part of the story. More than the knowledge, people want *personal understanding*. And in our credit-driven system they also want and need *theorem-credits*. (pp. 35–36)

Eventually, Thurston (1995) writes that:

> By concentrating on building the infrastructure and explaining and publishing definitions and ways of thinking but being slow in stating or in publishing proof of all the "theorems" I knew how to prove, I left room for many other people to pick up credit. There has been room for people to discover and publish other proofs of the geometrization theorem. These proofs helped develop mathematical concepts which are quite interesting in themselves, and lead to further mathematics. (p. 37)

Observe that Thurston's (1995) original approach closely resembles the axiomatic method described in Smith's (2006b) and Thurston's (1995) focus on "answers" suggests this. However, as his focus shifted toward "personal understanding," the field opened up so that unexpected

results could be achieved. These “ideal accidents” (Smith, 2006b, p. 145) places us closer to proof as problematics as opposed to axiomatics. Much like the manner in which Boards of Canada construct an album that produces a particular intensity in the way that tracks merge into one another, personal understanding is akin to this kind of intensity. A transition between tracks is an “ideal accident” just as Thurston’s (1995) loosening of his grip on the field of foliations allowed others to territorialize new avenues of study. Various forces (or series) are at play here. In the case of BoC, two tracks can be viewed as two series whose intersection (their collision) produces an event. In mathematical proof, the theorem result is the signified series and its proof is the signifying series. The two come together to form meaningful intensities that “lead to further mathematics” (Thurston, 1995, p. 37). Notice how Thurston (1995) explains how new proofs of a single theorem often lead to these new meanings. Rota (1997) also discusses how theorems have the potential to flow into new territories and that the result of a theorem may not be the true intensity of the proof. Therefore, given these notions, the collision of theorem/proof series is capable of opening up unexpected, accidental areas of mathematical thought well beyond the mere “fact” that is signified in the actual proof.

Thurston’s (1995) comment about “theorem-credits” is also extremely suggestive of the capitalistic posture that professional mathematics often takes on. Rotman (2000) further elaborates on this idea:

> Capitalism and mathematics are intimately related: mathematics functions as the grammar of technoscientific discourse that every form of capitalism has relied on and initiated. So it would be feasible to read the widespread acceptance of mathematical Platonism in terms of the effects of this intimacy, to relate the exchange of meaning within mathematical languages to the exchange of commodities, to see in the notion of a

> “timeless, eternal, unchangeable” object the presence of a pure fetishized meaning, and so on; feasible, in other words, to see in the realist account of mathematics an ideological formation serving certain (technoscientific) ends within twentieth-century capitalism. (p. 36)

This discussion on commodity explains Thurston’s (1995) ideas concerning the manner in which mathematical theorems can be seen as credits. The axiomatic approach to theorem proving not only foregrounds “answers” in the case of Thurston (1995), but it also foregrounds the “pure fetishized meaning” (Rotman, 2000) as opposed to meaning as an intensity capable of transmitting and flowing throughout the mathematics community without the blockages of an overly-territorialized (perhaps monopolistic) approach to obtaining “correct” proofs to theorems. In continuing to see how proof functions with BoC’s approach to musical assemblage, notice how a similar phenomenon occurs across their fan community. Rather than producing “closed” albums with clear, delineated tracks, BoC’s style is a method that has been taken up by other fans in assembling their own mixtapes consisting of various pieces of BoC material. The emphasis on transitions between tracks invites listeners to participate and construct their own albums with original transitions that produce new intensities despite the tracks themselves remaining mostly unchanged. The mixtape is the virtual, the plane of immanence for this community, that participates in the “ideal accidents” involved with problematics. BoC’s approach invites others to join in the creative process, to deterritorialize the musical space via the composition of new mixtapes. In doing so, BoC as “pure fetishized meaning” is disrupted (perhaps to the delight of BoC themselves) for an intensive meaning that deforms, reforms, and flows in multiple directions.

So far, I have shown that the virtual in both the context of Boards of Canada as well as in mathematical proof involves a community that is actively involved in shaping a landscape, whether mathematical or musical, to produce new intensities. These intensities involve a bringing together of forces (or series) that is the genesis of a musical work or a mathematical proof. Additionally, by engaging with these series in terms of problematics as opposed to axiomatics, the genetic aspects of these collisions are capable of destabilizing, though perhaps not eliminating, the capitalistic tendencies of commercial music as merely a way to extract a surplus value from consumers (in the case of BoC) or extracting "answers" and "theorem-credits" (Thurston, 1995) in the case of mathematical proof. Rather, in both the case of proof and BoC, there is an invitation for a community to participate in the growth of a particular field. The community, as opposed to any one individual, allows for produced intensities to flow across the body of proof. As Deleuze and Guattari (1980/1987) suggest, in terms of the Body without Organs (BwO):

> A BwO is made in such a way that it can be occupied, populated only by intensities. Only intensities pass and circulate. Still, the BwO is not a scene, a place, or even a support upon which something comes to pass. It has nothing to do with phantasy, there is nothing to interpret. The BwO causes intensities to pass; it produces and distributes them in a *spatium* that is itself intensive, lacking extension. It is not a space, nor is it in space; it is matter that occupies space to a given degree–to the degree corresponding to the intensities produced. It is nonstratified, unformed, intense matter, the matrix of intensity, intensity = 0; but there is nothing negative about that zero, there are no negative or opposite intensities. Matter equals energy. Production of the real as an intensive magnitude starting at zero. That is why we treat the BwO as the full egg before the

> extension of the organism and the organization of the organs, before the formation of the strata; as the intense egg defined by axes and vectors, gradients and thresholds, by dynamic tendencies involving energy transformation and kinematic movements involving group displacement, by migrations: all independent of *accessory forms* because the organs appear and function here only as pure intensities. The organ changes when it crosses a threshold, when it changes gradient. (p. 153)

In foregrounding proof as something that is a kind of flow that circulates throughout a creative community, what I am suggesting is that at the virtual level, at the level of the BwO, proof is not a singular, stable product that exists in textbooks and research articles and are often reproduced in the mathematics classroom. To suggest this is to say that proof, at its genesis, has extension. It does not. Rather, it is more akin to what D&G call a “spatium” or an egg (a body prior to its organization). Virtual proof, that is proof at its genesis, marked by key ideas that produce and are produced by intensities such as conviction, is prior to its organization in the axiomatic. The difficulty with virtual proof, then, is that while it is intensive and capable of engaging and producing event-effects, it is ultimately metaphysical, invisible, and without extension in the axiomatic. It functions as a BoC mixtape, capable of traversing and being traversed and producing and being produced. Virtual proof reaches thresholds that tip the balance of entire mathematical fields in unexpected directions as indicated by Thurston (1995). Sometimes, there is a sense that such thresholds have not yet been encountered but will be in the future as suggested in Rota (1997) concerning Fermat’s last theorem. Fermat’s last theorem provides a proposition, a mathematical fact, but the proof of this fact is far from stable or final: “The search that mathematicians are now beginning to perform upon the proof of Fermat’s last theorem is meant to disclose ‘what’ it is that is ‘really’ being proved” (Rota, 1997, p. 195). Virtual proof,

proof in its unactualized, unrealized form, is the plane of immanence akin to a machinic mixtape capable of being rearranged to produce new intensities and new meanings beyond what is implied in a textbook. In the next section, I continue to unpack this virtual, machinic side of proof by juxtaposing it with the notion of video game tool-assistance. By utilizing this type of analysis, I explore how proof is capable of events that collapse the relationship between signifier and signified series. This collapse allows for new intensities that are not in relation to mathematical meaning but, rather, attributed to the asignifying body of proof.

The Collapse of Sense

In constructing a BwO in relation to proof, I attempt to draw a comparison with analogous activities that feature aspects of the body and that could then be imported into the proof space. The activity I want to explore is that of video game "tool-assisted" speedruns. A *speedrun* involves the player purposely ignoring the expected manner in which one plays a video game in order to focus on how fast the game can be completed. For example, speedrunners often exploit glitches in the software in order to get to the end of the game as quickly as possible. Therefore, part of the activity of speedrunning involves finding ways to "break" the software in advantageous ways. As described in "Tool-assisted speedrun" (2024), a *tool-assisted speedrun* is a speedrun in which the controller inputs are programmed, rather than manually entered, often in order to get a sense of a game's theoretical minimum time to complete. With tool-assistance, one is not bound by the limitations of the human body to input wildly complex button combinations in a very short amount of time.

While tool-assistance is often done in the name of speedrunning, many individuals in this community utilize such technology as a way to explore the creative capability of the software rather than for simply understanding the shortest path to the end of a game. One particular run

appears to involve only programmed controller inputs, overloading the classic NES game *Super Mario Brothers 3* in order to gain "total control" of the software itself and reprogram it with new features such as a "color-a-dinosaur" activity and other graphical glitches that, though not intended to be a part of the game itself, were a latent potential in the software code (see Lord Tom, 2016). These reprogrammed features reflect a strange kind of poetry that are often involved with tool-assisted approaches to video games. With this type of tool-assistance, the signifier/signified relations between the game's code and the output become strained since this relation no longer resembles the original intention of the software. When total control is achieved via glitches, the resulting chaos appears to reflect Deleuze's (1960/1990) ideas concerning the collapse of sense. No longer does a game have organization, but rather, there is only schizophrenic desire. In this manner, even video games have bodies that are capable of resisting the signifier/signified relation that occur between a game's code and the resulting software that the end user experiences. I want to explore the signifier/signified relation and its collapse in video game speedrunning more closely and argue that mathematical proof functions similarly.

Concerning a video game, I focus on two particular series: 1) the underlying code that programs the game and 2) the resulting software that the player interfaces with. The program code operates in the sense of the proposition: variable declarations, strings of instructions, and even the desires of the programmers themselves are entangled in these lines of code. The code signifies the resulting piece of software (the second series) but the actual programming is invisible to the end user who only experiences the code's signified output. While the first series operates in the manner of the proposition, the second series operates in the manner of the state of affairs that pertain to the game itself (imagery, sound, controls, etc.). What is happening at the boundary between these two series? Recall that sense is the collision of both series so that both

sides matter in the formation of the sense event. From the side of the developer/programmer, the sense of the code is not the lines of programming itself but rather what the lines of code express: gameplay possibilities (among other things). On the side of the end user, it is the attribute of the state of affairs which also corresponds to gameplay styles. When these two come together, the result is convergence in the form of an engaging experience for the player and the game's coding operating efficiently in the way it was designed. There is also divergence in the form of the player engaging in unexpected play styles that may, for instance, generate glitches. The code determines these possibilities for the player, but the player also informs the game code and forces it to accommodate to actions that the developers, perhaps, did not anticipate. Therefore, there is a complex communication between signifying series and signified series that produce sense events during the operation of such software. Ideally, these sense events correspond to particularly engaging moments in the game. It is the goal of the developer to maximize the number of these events.

When utilizing tool-assistance, however, the signifying/signified relations between the code and the user software, or the sense event, becomes increasingly strained. The player pushes the experience to its limit, forcing the game code to accommodate the player in more and more extreme ways (e.g., overloading the game using rapid, programmed controller inputs) until there is a collapse when total control is achieved. This collapse no longer involves the signifying/signified relations between code and the user software (this is what has collapsed). Rather, the result is a world of the body without organs where glitches (paradox) become a strange, poetic form of desire. In the case of Lord Tom's (2016) tool-assisted speedrun of *Super Mario Brothers 3*, this collapse involves movements along a glitched digital landscape where, for example, the design of the levels has become scrambled and deterritorialized. Additionally, there

are moments of brief reterritorialization when the Total Control glitch allows for the software to be "reprogrammed" to involve brand-new game activities.

A similar example to such an approach to video games can be found in *Strawberry Cubes* by Schmidt (n.d.). In this particular game, the player controls an avatar (one that seemingly looks like Alice from Lewis Carrol's writings) in an abstract environment composed of glitching fractals, mysterious doors, strawberry seeds, hopping frogs, flying birds, and chasms that lead into deep, starry voids of violent sounds and light shows. All of the visuals are rendered mostly in various shades of red. The sound design is particularly striking: strange echoes accompany the sight of hopping frogs, glass dishes shatter with deep, loud rumbles, and the avatar that the player controls climbs ladders with high-pitched, squealing tones. This is a video game that has collapsed into the world of the body. Gone are the signifying relations found in the programming of *Super Mario Brothers 3*. In *Strawberry Cubes*, there is seemingly no end and no goal, only intensity and asignification. This particular example, like the previous one regarding tool-assistance, shows the latent potential in video games to become something that it was originally not intended to be: pure bodily desire that is free from the sorts of essentializing rules that typically accompany video games. In some sense, this bodily collapse becomes the world of the subject who engages with this type of interactive media. That is, the player loses their subjectivity and their sense of self. They "reach, not the point where one no longer says I, but the point where it is no longer of any importance whether one says I." Rather, they "have been aided, inspired, multiplied" (Deleuze & Guattari, 1980/1987, p. 3).

How does this relate to mathematical proof? In one sense, a computer program closely resembles that of a formal mathematical argument in that it is governed by logic structures, algorithms, and written statements that produce a piece of software as opposed to a new

mathematical theorem. Just as the lines of a mathematical proof signify its theorem, computer code signifies the user software. Like the code underlying a video game, the lines of a proof have the potential to signify a piece of "software" (a theorem) that can then be used and applied in mathematical research. When a proof for a theorem is obtained, the user of the theorem, often, is not interested in the proof itself but rather how the theorem can be used as a bridge to obtain new mathematical results. Therefore, the mathematician often wields a theorem in a similar manner to the gamer who wields the game controller. They are simply interested in the signified series, the end-user product, to pursue further experience. The signified series (the game code or proof) is invisible for these individuals. Both produce sense and meaning that result in gameplay styles in the case of video games or conviction and mathematical meaning in the case of a proof.

I demonstrated how video games have the potential for collapse into the body. How does this analogy carry over to proof? At a simple level, this can occur when there is an inability to grasp enough of a proof or theorem's proof-sense. Reading a proof, but being unable to connect all of the dots, can produce a painful bodily sensation. The resulting frustration is, perhaps, the beginning of the descent into the body where mathematical signification begins to fall apart. However, does the collapse into the body only occur during these intensely challenging moments? Does the body function even during mathematical successes? In de Freitas (2013a), the author explores a classroom episode in which a student utilizes a "pointer-thinger" to demonstrate his thinking about a particular mathematical problem involving the number of squares found in a 6x6 grid. De Freitas (2013a) writes:

> In the case of Colin, language fails to operate as a form of communication (of his thinking). He uses a series of indexical spatial references ("one side", "bottom", "it's ten", "that was") that bind his speech to the actual diagram on the overhead projector.

> Colin's stammering troubles the regime of communication that would demote his thinking to some entirely linguistic form, for as Lecercle (2002) suggests, "the speaker is in constant danger of being burked by language: a wet blanket of signification smothers any attempt at expression" (p. 6). When the teacher asks that Colin go up and show the others what he means, he confidently asks if he can use the teacher's "pointer-thinger" to engage the diagram. In so doing, a new material assemblage is formed (Colin-"pointer-thinger"-diagram) that functions to de-center language as communication and leverage it instead as expression. (p. 135)

Further on, de Freitas (2013a) writes:

> In the case of Colin, we see how his stammering re-assembles the content-expression links, that the sounds he makes are asignifying particles or intermediate entities that, together with the hands, the projector, the diagram, and various other material-affective forces, produce an entirely new mathematical assemblage (p. 137)

Like tool-assistance in video games as described earlier, language (or code) breaks down here and the embodied assemblage of student-pointer-diagram begins to dominate. The emphasis on the pointer animating the diagram puts us closer to the experience of *Strawberry Cubes* where the bodily world of desire overtakes any type of meaning or signification. This is where the collapse of sense begins. That's not to say that signification is not a part of mathematics or experimental video games at all, but only that, at the genetic level, becoming is more a matter of intensity than it is about meaning, involving "an ontogenetic force or flow of energy that activates as much as it is activated" (de Freitas, 2013a, p. 138) and that decenters the expectations of traditional video games as well as traditional, static approaches to mathematical reasoning. Genesis is always about the production of the new, whether it is mathematics as the

animation of diagrams or video games as a site of psychedelic wanderings as opposed to clear rule sets and traditional gameplay styles. Concerning this aspect of the collapse into the body, Deleuze (1969/1990) writes:

> In this collapse of the surface, the entire world loses its meaning. It maintains perhaps a certain power of denotation, but this is experienced as empty. It maintains a certain power of manifestation, but this is experienced as indifferent. And it maintains a certain signification, experienced as "false." Nevertheless, the word loses its sense, that is, its power to draw together or to express an incorporeal effect distinct from the actions and passions of the body, and an ideational event distinct from its present realization. Every event is realized, be it in a hallucinatory form. Every word is physical, and immediately affects the body. The procedure is this: a word, often of an alimentary nature, appears in capital letters, printed as in a collage which freezes it and strips it of its sense. But the moment that the pinned-down word loses its sense, it bursts into pieces; it is decomposed into syllables, letters, and above all into consonants which act directly on the body, penetrating and bruising it.…The moment that the maternal language is stripped of its sense, its *phonetic elements* become singularly wounding. The word no longer expresses an attribute of the state of affairs; its fragments merge with unbearable sonorous qualities, invade the body where they form a mixture and a new state of affairs, as if they themselves were a noisy, poisonous food and canned excrement.…In this passion, a pure language-affect is substituted for the effect of language." (pp. 87–88)

I argue that what Deleuze (1969/1990) is describing here is precisely what is happening in both *Strawberry Cubes* as well as the episode described in de Freitas (2013a) involving the student-pointer-diagram assemblage. That is, at the level of genesis, mathematics and game are

"decomposed" into the noisy movements of pointers, diagrams and player avatars, crashing glass dishes, and glitching light shows. Words as language no longer carry meaning or sense, only wounding effects that form a genesis, a problematic, where solutions and resolutions are stumbled into rather than correlated or actively pursued.

Another example of such a mathematical episode, more specifically involving proof this time, can be found in a chapter from the book *Living Proof: Resilience Along the Mathematical Journey*[1] published by the American Mathematical Society. In this particular chapter, Dr. Robert Allen describes a scenario during the end of his doctoral studies involving a difficult proof result that had been challenging him for a while.

> This was causing me to lay awake at night thinking about how this result was going to keep me from earning my PhD. In times like these, a long hot shower usually relaxes me enough to fall asleep.…As I am standing under the scalding water trying to wash away the anxiety and frustration of my situation, a statement/question pops in my head. "I *must* know something about compact operators. What do I know about compact operators?'' The answer was three words: The Spectral Theorem. That's right, the glory of every functional analysis class on the planet. By this time, the steam had covered the shower door, creating a wonderful writing surface. So, I started writing the Spectral Theorem. I then feverishly began to connect the dots, and, in the fog of a steaming shower, I wrote the cutest result in my dissertation. I immediately jumped out of the shower to grab a piece of paper and a pencil to jot this idea down. Yes, I was running around my apartment in my birthday suit, dripping wet, giggling like a preteen watching *Twilight* for the 18th time (Team Jacob). Once the adrenaline worked its way through my system and

[1] Excerpts reprinted with permission from the American Mathematical Society and their respective authors.

> I had put some clothes on, I sat down to write the results and send them to my advisor, proclaiming victory! Then all the self-doubt flooded back in the single thought: "What if there is something wrong with my proof that I can't see?"…Fortunately, this story has a happy ending. The proof was correct. (Allen, 2019, p. 80)

Like the situation described in de Freitas (2013a), Dr. Allen's relationship to proof does not only involve the access of a proof's sense or "key idea" (Raman, 2003). Rather, it involves an assemblage between steam, water, shower wall, bodily affect, and key idea expressed in the writing on the shower wall. Dr. Allen is in the world of *Strawberry Cubes*. Rather than crashing dishes, there is the sounds of rushing shower water and perhaps the squeaking of Dr. Allen's finger on the shower wall. Rather than the environment of the mathematician's office, there is the hazy environment of a steamy shower. These aspects are akin to "Collin's stammering" that "troubles the regime of communication that would demote his thinking to some entirely linguistic form" (de Freitas, 2013a, p. 135). These relations strain the signifying/signified relations that exist between theorem and proof just as the strange poetry of tool-assistance and of *Strawberry Cubes* strains the relations between code and its software output. The strain leads to a collapse where proof is now a matter of bodily intensity in the form of a joyous romp through Dr. Allen's apartment. No longer is Dr. Allen himself. Rather, he is in the throes of an animalistic desire that serves as a genesis to the mathematical knowledge that he is constructing.

It is the above discussion concerning desire, body, and collapse that corresponds to Rota's (1997) comments that a proof's "value" is "not in what it proves, but in what it opens up, in what it makes possible" (p. 191). Thurston's (1995) discussion of the field of foliations is a similar example; by loosening his grip on the field and the rigorous manner in which he produced his results, this particular mathematical area was able to open up so that new theorems

and proofs could be made possible. In both cases, there is a commonality: deterritorialization. When proof becomes deterritorialized toward the virtual, the focus is no longer that of axiomatics, answers, and what a proof seems to prove. Rather, it moves toward a BwO which "is what remains when you take everything away" (Deleuze and Guattari, 1980/1987, p. 151). What is being taken away? Deleuze and Guattari (1980/1987) elaborate:

> What you take away is precisely the phantasy, and significances and subjectifications as a whole. Psychoanalysis does the opposite: it translates everything into phantasies, it converts everything into phantasy, it retains the phantasy. It royally botches the real, because it botches the BwO. (p. 151)

Much of Deleuze and Guattari's (1980/1987) discussion of the BwO appears to be in opposition to Freudian psychoanalysis. However, one can replace references to "psychoanalysis" and "phantasy" with "proof" (in the traditional sense) and "axiomatics" respectively and a very similar discussion could commence: traditional notions of proof, like psychoanalysis, is a territorialization of the BwO. It forces meaning and solutions rather than allowing for an opening up of intensities. At its genesis, at the level of the virtual, proof is intensity. It flows through the cracks that partitions the axiomatic dimension of mathematics, allowing for the possibility for new territories of mathematical thought to flourish. It is the world of the mixtape as well as that of the "broken" video game. It can be difficult to speak of since, as mathematicians, we do not have direct access to virtual proof like we do with "actual" proof. Rotman (2000) in his book *Mathematics as Sign: Writing, Imagining, Counting*, appears to provide a theoretical model that takes into account this virtual space: that of the *Virtual Code* which is

> the domain of all legitimately imaginable operations, that is, as signifying possibilities available to an idealization of the Subject. This idealization, the one-who-executes these

> activities, the *Agent*, is envisaged as a surrogate or proxy of the Subject, imagined into being precisely in order to act on the purely formal, mechanically specifiable correlates–signifiers–of what for the Subject is meaningful via signs. (p. 52).

This virtual aspect of proof functions as a kind of stream or flow that is capable of moving in many directions at once. At this virtual level, this code is intensive and cannot easily be measured as opposed to the actual lines of a proof that often communicate clear meaning or direction. The virtual is the "opening up of possibilities" that Rota (1997) envisions. What Rotman's (2000) model does not appear to take into account, however, is the assemblage aspect of virtual proof. The virtual code functions with other elements of a larger machine, even those that may not seem to be related to mathematics as was seen in Allen's (2019) story. As an analogous example, consider Buchanan's (2013) thinking on D&G's analysis of Little Hans who was a boy that Freud assisted in treating after it was found that he had an intense fear of horses:

> Deleuze and Guattari move away from Freud by shifting the domain of analysis from representation to affect, which still has its semiotic dimension, to be sure, but it is a matter of aggregated signs (populations) rather than ready-made symbols. The horse no longer stands for something other than itself (i.e., it no longer represents Daddy); it is now the aggregated sign of a particular kind of "feeling." That feeling isn't defined by "horsiness" or the sense that one is somehow horse-like; rather, it is defined by the affects that in a particular assemblage are associated with horses, such as having one's eyes blocked, being restrained with bit and bridle, the sense of pride one is nevertheless able to maintain in spite of such restraints, and so on. It is these attributes of a horse's working life in 19th-century Vienna that resonate for Little Hans, not the fact of its existence. Deleuze and Guattari use the term "affect" to designate these feelings because they occur

> at a level beneath or perhaps before ideation. If they call Hans's feelings "becoming-horse," it isn't because Hans is thinking about horses or is in danger of becoming one, but rather because the affects he is experiencing are those we associate with horses, such as being restrained. (p. 16)

In Allen's (2019) story, it is not the case that the intensity of the proof he struggles with resides only in the quality of the key idea or the sense that serves to signify a solution (i.e. axiomatics). Rather, it is the "aggregate" of such affects (the shower wall, steam, key idea, water, etc.) that constitutes proof as a becoming. Those who construct or get caught up in such an assemblage, like Dr. Allen, can experiences this. Therefore, when a student or expert of mathematics struggles with a proof as they scribble on a chalkboard, or a shower wall for that matter, one becomes entangled with a virtual genesis. This could manifest itself as a kind of pain or suffering, a BwO, that only circulates these affects, and of which one may never escape from as is sometimes the case with struggling mathematics students. However, such constructions could also result in a BwO where the student or mathematician is a part of an "opening up of possibilities" (Rota, 1997, p. 191). This sort of BwO, a *full* one as Deleuze and Guattari (1980/1987) call it, is capable of the free transmissions of intensities without blockage. This is the BwO that Dr. Allen constructs as he develops his shower proof. It is also the BwO that is constructed after Thurston (1995) loosened his grip on the field of foliations, deterritorializing proof into the virtual where only intensive possibilities reside. But with each construction, there is also collapse. This collapse involves the destruction of sense and meaning. In Allen's (2019) case, the key idea that he is able to leverage while showering eventually breaks down and collapses into the body. Dr. Allen runs around his apartment, in his "birthday suit," celebrating his victory and making contact with a virtual genesis. No longer is proof about its sense and the

key idea. Rather, it is about the intensities that transmit and circulate along the contours of a virtual proof "spatium" that exist in parallel to its actual "axiomatic." These intensities cannot transmit within the written proof itself but can only do so with the help of other assemblage components such as other mathematicians, materials (e.g., a shower wall, pointing devices), actions (e.g., running, hiking), and affects. This intensive body, however, also runs parallel to its axiomatic form in the actual. As Deleuze and Guattari (1980/1987) state:

> One side of a machinic assemblage faces the strata, which doubtless make it a kind of organism, or signifying totality, or determination attributable to a subject; it also has a side facing a *body without organs*, which is continually dismantling the organism, causing asignifying particles or pure intensities to pass or circulate, and attributing to itself subjects that it leaves with nothing more than a name as the trace of an intensity. (p. 4)

The platonistic axiomatics of proof, with its focus on solution, mental processes, cognition, and imitation of the ideal, play an important role in mathematics. However, the problematics of proof, the virtual side, is intrinsically tied to axiomatics while also *not resembling or signifying it*. Therefore, virtual proof, or *becoming-proof*, cannot be ignored as it serves as the genesis of the axiomatic, the platonic, and all of mathematics itself.

Becoming-Proof

As I have discussed in this chapter, the philosophy of Gilles Deleuze and Félix Guattari offers a new way to see mathematical proof as a reversal of platonism where the difference of virtual relations is primary over (or at least existing alongside) the traditional stability of identity and sameness. Therefore, while still maintaining a connection to these notions, virtual proof will not "look" the same as the sorts of characterizations given by those who see proof as primarily a

problem-solving activity where the mental processing of the individual or community is foregrounded. Proof is also not an objective process for discovering a theorem that resides in a transcendent plane as a kind of abstract object awaiting discovery. To take these perspectives is to see proof as belonging only to the axiomatic and the actual. Virtual proof is a deterritorialized BwO where only the contours of an axiomatic proof remain, consisting of intensities and effects (such as conviction) and collisions and events (such as the key idea). We can begin to see this virtuality by plugging the machine of proof into other machines and observing what is produced. What we witness is that proof functions with mixtapes, video games, showers, as well as with other individuals. Proof is not a stable identity but, rather, an irreducible assemblage that is always in function with other components. This assemblage is the BwO. Deleuze and Guattari (1980/1987), referencing Artaud (1977), write: "*The body is the body. Alone it stands. And in no need of organs. Organism it never is*" (p. 158). Proof as BwO allows for intensities to transmit (but not axiomatic organs) only when it can function with other machines (shower, music, software, other people, etc.). As more intensities flow, an axiomatic proof begins to develop, and a *becoming proof* occurs. This can be seen in both Dr. Allen's (2019) case as well as Thurston's (1995). Both needed to let go of the actual so that contact could be made with the virtual body and a proper genesis for their work could be formed. What we are left with, then, is a theory of proof consisting of collisions, events, effects, and intensities: a sort of *physics of proof* (or metaphysics) rather than a cognitive or axiomatic theory of expert mathematical thought.

Reaching the Plane of Immanence: The Intensive Genesis of Proof in Expert Mathematical Practice

In the mathematics education literature, proof is often seen as a cognitive process (i.e., a problem-solving activity), where student and expert personal and social reasoning are foregrounded (Stylianides et al., 2017). For example, Weber (2001) investigates how individuals' *strategic knowledge* or "heuristic guidelines that they can use to recall actions that are likely to be useful or to choose which action to apply among several alternatives" (p. 111) plays an important role in proving among more expert mathematicians. Raman's (2003) notion of the *key idea* suggests that expert provers begin in a private, informal domain of problem solving toward a central idea that allows them to pivot into a more rigorous, formal, public proof presentation. Harel and Sowder (1998) place importance on students' proof schemes, a kind of frame that is used to gain personal certainty regarding an argument such as appeals to empirical evidence, where one becomes convinced of a general mathematical assertion based on a small number of successful cases, and authoritarian evidence, where one becomes convinced of an assertion if a trusted source (such as a teacher) supports the claim. Weber et al. (2014) suggest that expert mathematicians also gain conviction from these types of evidences as well.

While these investigations are certainly important, they often situate proof within the problem-solving subject (or between multiple subjects) where ideal mathematical results are to be approximated or imitated. Smith (2022), following Deleuze (1969/1990), would seem to suggest that such a perspective of proof is *static* in that mathematical meaning or *sense* is a question of adequating a mathematical states of affairs (the mathematical statement to be proven) with our reasoning about such affairs (the proof process). Sense, the "real" proof, is not located in the sentences of the written argument nor in the actual state of affairs but in the sterile

meaning that results from a proof or theorem. In this case, proof is seen as something that naturally follows from already formed statements. When reading a proof, understanding (the "real" proof) occurs because written statements have guided our thinking. Here, statements are primary and our understanding or sense is the result of these statements. A similar argument can be made when writing a proof for a given statement. I begin with a statement to be proven and this statement suggests possible directions for us to investigate. That is, the sense of the proof is secondary to the theorem statement. Citing Deleuze (1969/1990), Smith (2022) indicates that sense might also be seen as "the *condition* of truth" (p. 9). In In the case where sense is seen as the condition (rather than the result) of fully-formed statements, sense (the "real" proof) is something that occurs prior to mathematical propositions and written arguments. After all, some sort of understanding or meaning would have to have been achieved in order for the formation of any formal statements (whether it is an actual proof or a theorem statement) to occur in the first place.

This issue of order or primacy of sense over the proposition (or the other way around), is important since the mathematics education literature, as discussed above, appears to take a side on this issue. By foregrounding cognitive processes and mathematical meaning (as discussed earlier), I argue that the literature pertaining to mathematical proof often sees sense as the condition for a fully formed proof. The important idea is that, in our application of Deleuze's (1969/1990) notion of static genesis to the mathematics education literature, sense is what allows a proof argument to correspond to its theorem statement. If such an adequation were not possible, then mathematical proof would not be possible. That is, sense must be possible in order for a formal proof to be possible. In this line of thought, a proof's sense (or *proof sense*) is the necessary condition for a theorem to be proven. As Smith (2022) points out, however, it is this

conditional aspect of sense that also renders it static. What this *static genesis* of proof fails to take into account is how mathematical meaning is *produced* rather than simply serving a *signifying* function grounded in sense. That is, as a way to pass from a proof argument to the theorem statement (or the other way around). In this paper, I are not as concerned with proof's signifying function since it appears that much of the mathematics education literature on proof connects to this aspect already. Rather, I am interested in investigating proof's existence *beyond* the signifying and into the *asignifying*. I seek to explore what this existence looks like and its importance to the mathematician. More specifically, I investigate Deleuze and Guattari's (D&G) thinking concerning the ways in which the body produces mathematical meaning in the context of an expert proof situation. By focusing on the asignifying body, Deleuze (1969/1990) shows how meaning is produced immanently. I apply this framework to the study of how experts engage with mathematics and demonstrate that proof emerges not from a grounded, axiomatic foundation that places sense and the mathematical subject as the arbiter of mathematical meaning (the signifying function), but from a groundless, asignifying genesis of becoming.

Furthermore, I demonstrate and apply this framing through an analysis of an expert proof story written by Neel (2019) from the book *Living Proof: Stories of Resilience Along the Mathematical Journey*[2] published by the American Mathematical Society. This story is an autobiographical account of a proof experience that the author had near the end of his dissertation work and stands in stark contrast to the notion that proof is "grounded in" or begins with the sense of the subject. Rather, Neel's (2019) story demonstrates that mathematical proof is not merely the result of cognitive or socially constructed knowledge but rather the result of a different kind of construction: that of an assemblage that exists outside of the self as well as

[2] Excerpts reprinted with permission from the American Mathematical Society and their respective authors.

outside any sort of transcendent, ideal space. These assemblages are united not by any sort of essence, but by material forces, or intensities, that flow, move, and transmit between the elements of the assemblage. This assemblage then has the power to engage in a sort of synthesis, production, or becoming in which a fully-formed proof is actualized.

Before sharing my analysis of Neel's (2019) story, I begin by providing some further necessary background regarding the post-structural philosophy of Deleuze and Guattari (D&G), starting with their notion of the plane of immanence which serves as a kind of reversal of idealist platonic thought. Then, I discuss the concepts of sense and event followed by intensity and the body without organs (BwO). After presenting the analysis of the story material, I conclude the paper with a discussion of some broad implications for the teaching and learning of proof.

Plane of Immanence

Gilles Deleuze and Félix Guattari were both French philosophers in the post-structural tradition, one that reacts to the essentializing nature of structuralism which often stresses stable identities, assumptions, and metanarratives. For example, in mathematical practice, axioms (statements that cannot be proven but are simply taken for granted) form the basis for engaging in mathematical activity, with entire fields deriving from just a handful of such assumptions. This positions mathematics in the world of *platonic form* where "facts" are transcendent ideas to be "discovered" or approximated in reality, even if this reality merely exists in the minds of mathematicians. These unconditioned axioms are what provide mathematics with its stability. A proof is true because mathematicians begin with an axiomatic method that secures its results in the transcendent. Where is this transcendence located? Rotman (2000) notes that it is located in the "eternal" and that "real" mathematics is meant to "mimic" this ideal, objective space. Rotman (2000) also discusses mathematical *intuitionism* which positions as primary the subjective

construction of mathematics. Here, eternal mathematical truth is replaced with mental constructions that determine mathematical "fact." This approach suggests that if people were to disappear, so would mathematics since this sort of activity ultimately concerns the structure of the human mind. Here, transcendence is located not in an external, ideal reality but within the subject. D&G resist such conceptualizations of thought and knowledge by formulating a theory that avoids becoming ensnared in such transcendent relations. Rather, they opt for a foundation that commits to *immanence* which is the notion that seemingly universal values and properties are not separate from and "above" the world but are always within and connected to it. As described by Philp Goodchild regarding immanence: "It's not the case that thought is within us, but thought is a kind of environment that we enter into and are already within" (Timeline Theological Videos, 2014, 9:09). In this dissertation, I view proof in a similar manner.

For D&G, this "world" does not directly refer to some literal or physical reality that people find themselves in but rather a world that is always to be found at the edges and the extremes of life. Such a world, then, is difficult to encounter since it requires radical thought that is at odds with notions pertaining to everyday "common sense." However, unlike transcendence, it is not impossible to have an encounter with this world which Deleuze and Guattari (1991/1994) call, *the plane of immanence*. This notion is challenging, but D&G do provide some important hints for how to make sense of this plane. For instance, Deleuze and Guattari (1991/1994) refer to the plane of immanence as "the image of thought, the image thought gives itself of what it means to think, to make use of thought, to find one's bearings in thought." (p. 37). Deleuze (1995/1997) is also adamant that the plane of immanence is only immanent to itself:

> it is not in something, not *to* something; it does not depend on an object and does not belong to a subject. In Spinoza, immanence is not immanence *to* substance, but substance and modes are in immanence.…it is when immanence is no longer immanence to anything other than itself that we can talk of a plane of immanence (p. 4).

This ensures that there is never a "higher system" in which the plane is dependent on and which only serves to "revive," as Deleuze and Guattari (1991/1994) put it, transcendent reality. But if there are no transcendents, then what is it that remains? Deleuze's (1995/1997) response: "sheer power, utter beatitude" (p. 4).

Deleuze and Guattari (1991/1994) indicate that the plane of immanence resembles a kind of mean or consistency, writing:

> The problem of philosophy is to acquire a consistency without losing the infinite into which thought plunges (in this respect chaos has as much a mental as a physical existence). *To give consistency without losing anything of the infinite* is very different from the problem of science, which seeks to provide chaos with reference points, on condition of renouncing infinite movements and speeds and of carrying out a limitation of speed first of all (p. 42).

Therefore, the plane of immanence, while requiring a degree of extremism, never pushes far enough that it loses itself. That is, the plane always retains a consistency while maintaining the infinite that is key to thought itself. Science and mathematics often do not support this as it sacrifices the infinite for practicality, stability, power, and capital. Now, this does not mean that science and mathematics do not have a plane of immanence. They do. However, it is often not reached due to the interests of the State and of capitalism. Deleuze and Guattari (1980/1987) suggest that the plane of immanence may "flow" with State interests but not in a manner that

serves to further reify their structures. Rather, this flow serves a *deterritorializing* function, one in which the *possibilities* of science is made primary over its stability. This possibility also introduces the potential for science and mathematics to engage in decolonizing efforts rather than supporting capitalistic interests that are often associated with the very industries that benefit the most from scientific and mathematical activity. The plane of immanence is thought at its genesis which involves only movement, intensity, and affect. That is, the plane of immanence is the physics or the body of thought rather than that which is contained within the subject.

There is one more aspect of the plane of immanence that is worth discussing. Wolfendale (2009) notes that D&G differentiate between *a plane of immanence* and *THE plane of immanence*. The former relates to what I have just discussed which is an epistemological notion, or "power of thinking" (Deleuze and Guattari, 1991/1994, p. 48), regarding how knowledge is always in the process of being constructed and never in a stable state: an ungrounded ground of thought. Deleuze and Guattari (1991/1994) write that "Philosophy is a constructivism, and constructivism has two qualitatively complementary aspects: the creation of concepts and the laying out of a plane" (pp. 35–36). A mathematician may begin by assuming a set of axioms that are then capable of generating entire fields, but the ground of these fields are not axioms but rather "the horizon out from which thinking as such can take place, and thus constitutes the internal condition of thinking" (Spindler, 2010, p. 151). Here, the plane of immanence is given as an epistemological explanation for the construction of mathematics.

However, when Deleuze and Guattari (1991/1994) discuss *THE* plane of immanence (also called the *pure plane of immanence*), or "power of being" (p. 48), they are actually referring to something more closely associated with Spinoza's theory of substance. That is to say, *THE* plane of immanence is the ontological field that connects everything together (whether

it be ideas, physical matter, cognitive process, etc.) in a singular material. Deleuze and Guattari (1991/1994) write that *THE* plane of immanence is “the nonthought within thought” (p. 59). *THE* plane of immanence is constituted by causal affects and intensities that allow for change and becoming. For example, the relation between a proof and its theorem is not simply that of logical deduction but also of a mathematical community, technology, as well as physical allowances (office space, conference events, etc.). As Wolfendale (2009) notes, even “transmissions of sound, light and pheromones” (para. 28) connect what is on *THE* plane of immanence, as these intensities are needed in order to properly interact with others and the world around us. What this means, according to Wolfendale (2009), is that becoming does not simply involve familiar causal relations. In the case of proof, becoming is not just a matter of logic but also a matter of particles, personal relationships, and perhaps even of ethics. All of these aspects, while not all logical relations, contribute toward what I call *becoming-proof*. This becoming involves the image of thought (a plane of immanence) as part of *THE* plane of immanence. Deleuze and Guattari (1991/1994) write

> It is the base of all planes, immanent to every thinkable plane that does not succeed in thinking it. It is the most intimate within thought and yet the absolute outside–an outside more distant than any external world because it is an inside deeper than any internal world….Perhaps this is the supreme act of philosophy: not so much to think *THE* plane of immanence as to show that it is there, unthought in every plane, and to think it in this way as the outside and inside of thought, as the not-external outside and the not-internal inside–that which cannot be thought and yet must be thought….Spinoza, the infinite becoming-philosopher: he showed, drew up, and thought the “best” plane of immanence–

> that is, the purest, the one that does not hand itself over to the transcendent or restore any transcendent. (pp. 59–60)

When discussing the plane of immanence, I will not always distinguish between *a* plane of immanence and *THE* plane of immanence since they are both linked in the manner given by D&G above. When discussing becoming in the manner of Spinoza, that everything is part of a single substance, *THE* plane of immanence is in reach. *THE* plane of immanence allows for connections to be made between mathematical proof and other types of material under a single field. When I discuss proof as an epistemological endeavor, as a method for the construction of mathematical knowledge, I refer to a laying out of a plane that serves as the unstable ground for proof to occur but that is also connected with all other material on *THE* plane of immanence.

Later, I will demonstrate how certain relations, which are seemingly outside of mathematics, contribute toward the virtual proof processes that make up a proof while contributing to *THE* plane of immanence itself. In this paper, I argue that encountering *THE* plane of immanence through proof puts the mathematician in a position to better obtain results and also produces and fosters a disposition or ethic, one that is cosmic rather than capitalistic. Such a disposition or ethic situates the mathematician to become an instructor of proof more capable of meeting the needs of students over the needs of the State and industry.

In order to better understand the plane of immanence as it relates to proof, it is necessary to provide more background regarding the concepts and ideas that populate and “flow” within it. In the next section, I discuss the concepts of sense and event in the context of the static genesis, which sets up the conversation about proof as becoming and the resulting productions from such processes.

The Static Genesis of Proof

In *The Logic of Sense*, Deleuze (1969/1990) introduces his notion of *sense* as it pertains to language. Forces, or what Deleuze calls *series*, collide with one another to form a kind of *sense* or *event* that induces and is the genesis for meaning. Consider the following example from Smith, Protevi, and Voss (2023):

> I can attribute the proper name "Battle of Waterloo" to a particular state of affairs, but the battle itself is an incorporeal event (or sense) with no other reality than that of the expression of my proposition; what we find in the state of affairs are bodies mixing with one another—spears stabbing flesh, bullets flying through the air, cannons firing, bodies being ripped apart—and the battle itself is the *effect* or the *result* of this intermingling of bodies. (3.2 *Logic of Sense* section)

Here, "Battle of Waterloo" involves two series: 1) the proposition "Battle of Waterloo" itself and 2) the actual state of affairs of the battle. The actual "battle" is not located in the material "bodies mixing with one another" (Smith, Protevi, & Voss, 2023, 3.2 *Logic of Sense* section) nor is it found in the words I use to name or describe it but rather in the *expression* of the proposition that is also *attributed* to the actual state of affairs (the mixing of bodies). Deleuze (1969/1990) writes "*Sense is both the expressible or the expressed of the proposition, and the attribute of the state of affairs*.…It is exactly the boundary between propositions and things" (p. 22). The collision between propositions and states of affairs allows for enough coherence for meaning to be made while also introducing an element of divergence, since states of affairs and propositions are fundamentally different phenomenon. There will always be a degree of disequilibrium in the process of meaning-making. Therefore, according to Brady (2017), "sense is all that is holding words and objects apart, *and* is all that is holding them

together" (Reversing Sense section). Deleuze (1990) refers to this type of sense (the expression of the proposition adequated with the attribute of a corresponding state of affairs) as the *static genesis* of language. Hughes (2008) elaborates:

> This genesis begins in the empty form of time rather than in the dispersion of the material object in movement (as in the dynamic genesis of *Anti-Oedipus* and *The Logic of Sense*) and ends in the representation of things in general for a human consciousness. It is thus a static rather than a dynamic genesis. (p. 120)

Now, with regards to the static genesis, consider the example of mathematical proof. How do I get from a proof to its result (or theorem)? Similar to the example above concerning the battle, a proof is more than just a series of deductions memorialized in text. That is, proof is not merely located within a mathematical text but rather found in its sense. For example, consider Raman's (2003) notion of the *key idea*, which

> is an heuristic idea which one can map to a formal proof with appropriate sense of rigor. It links together the public and private domains, and in doing so gives a sense of *understanding* and *conviction*. Key ideas show *why* a particular claim is true.…The connection is rather straightforward: the heuristic idea is essentially private, the procedural idea is essentially public, and the key idea provides the link between the two (pp. 323-324)

Here, the key idea is a heuristic that connects both informal, private domains (how one is *personally* convinced by an argument) with a more formal, rigorous presentation (what might be seen in a published proof). The key idea does not reside in one's private reasoning nor in its corresponding state of affairs as a publicly proven "fact" but, rather, it is in the sense that adequates these two series. Kidron and Dreyfus (2014) developed a similar concept known as a

proof image which represents an individual's visualization of an argument that precedes a formal presentation. Harel and Sowder (1998) suggest that it is a proof's subjective heuristic, or *proof scheme*, that ultimately produces certainty and, therefore, should be stressed in the teaching and learning of proof:

> we present proof of well-stated, and in many cases obvious, propositions, rather than ask for explorations and conjectures. As a consequence, students do not learn that proofs are first and foremost *convincing* arguments, that proofs (and theorems) are a product of human activity, in which they can and should participate; that they are an essential part of doing mathematics. This is in essence the whole thrust of our teaching treatments. The goal is to help students refine their own conception of what constitutes justification in mathematics: from a conception that is largely dominated by surface perceptions, symbol manipulation, and proof rituals, to a conception that is based on intuition, internal conviction, and necessity (p. 237).

The idea that this literature supports is the existence of mental constructions (that is, sense) that are (or should be) *primary* to rigorous proof formulation. Therefore, this literature supports an approach to proof that is consistent with Deleuze's (1969/1990) notion of static genesis. Notions such as the key idea, proof scheme, and proof image integrate forces (or series) that often involve some form of informal, private thought with more formalized, public, and rigorous presentation. There is an emphasis on linkage here and an *event* which has a before and after. The individual engages in private, informal thought which eventually gives way to a heuristic event that then allows for the actualization of a rigorous, complete proof. This heuristic event, or what I refer to as a *proof event*, is akin to Deleuze's notion of the static genesis of sense as described at the beginning of this section. These events allow for a passage toward a rigorous, public proof while

not being fully complete and consistent. Therefore, proof-sense involves both convergence and divergence of series. If divergence did not exist alongside convergence, there would be nothing to notice and no mathematical meanings to signify in the first place. This integration is the transcendent, incorporeal, sterile effect (the key idea, the proof image, etc.) that serves as the condition for comprehending an already completed written proof or the construction of a new original proof argument.

The Dynamic Genesis of Proof

The previous section describes how the static genesis of proof involves series that both converge *and* diverge to form mathematical meanings or proof-events. These events, while seemingly coherent, are also not in and of themselves complete and consistent. More is needed beyond the key ideas, proof schemes, etc. that contribute toward the *sense* of an argument. These heuristics, however, provide the genesis for actual, rigorous proofs. In terms of proof, sense is the *boundary* between the signifying world of the proof argument and its signified state of affairs as mathematical "fact," providing the necessary link between the two worlds. However, this notion of sense tends to place a strong emphasis on semiotics, language, meaning, and signification as well. The proof-sense that connects the two series deals in the effects and attributes that ultimately signify new mathematics, whether it is in the comprehending of an already completed proof or in the obtaining of a new, original mathematical fact or result. However, the main concern of Deleuze's (1969/1990) *The Logic of Sense* is that of the *dynamic genesis*, which indicates that sense and the event are *actively produced* from the chaos of the desiring, asignifying body rather than the result (or the condition) of an association between expressions and attributes of propositions and state of affairs. Smith (2006a) appears to suggest that Deleuze experienced a change after *The Logic*

of Sense in writing his next book *Anti-Oedipus*, which was written with Fèlix Guattari. No longer was Deleuze interested in sense but rather the body. Smith (2006a) writes:

> Whereas *Logic of Sense* was content to remain at the surface of sense (Lewis Carroll), *Anti-Oedipus* can be said to have plunged into the depth of bodies (Artaud). Why, then, does Deleuze no longer speak of the "depths" in *Anti-Oedipus*? At the very least, the concept of depth has relevance only from the viewpoint of a theory of surfaces; outside of that context, the notion of depth loses its relevance. Once Deleuze ensconces himself in the depths, so to speak, he requires a new conceptual apparatus. This is why in *Anti-Oedipus* the surface-depth problem of *Logic of Sense* is replaced with the problem of the body without organs, and the flows that traverse it. (p. 147)

In *Anti-Oedipus*, D&G were not interested in how the surface (sense) can give way to the materialist world of the body. Rather, D&G explore the body on its own without the signifying world of sense to guide it. D&G see the body as *asignifying*, that is, without a meaning attached to it or a stable identity to imitate. This body, then, becomes what D&G refer to as the *body without organs* (BwO). In their book *A Thousand Plateaus*, Deleuze and Guattari (1980/1987) write: “The BwO is opposed not to the organs but to that organization of the organs called the organism” (p. 158). They also liken the BwO to an egg in that it is the unorganized, undifferentiated organism with the potential for various becomings. How are these becomings facilitated? Deleuze and Guattari (1980/1987) write:

> A BwO is made in such a way that it can be occupied, populated only by intensities. Only intensities pass and circulate. Still, the BwO is not a scene, a place, or even a support upon which something comes to pass. It has nothing to do with phantasy, there is nothing to interpret. The BwO causes intensities to pass, it produces and distributes them in a

> *spatium* that is itself intensive, lacking extension. It is not a space, nor is it in space; it is matter that occupies space to a given degree–to the degree corresponding to the intensities produced. It is nonstratified, unformed, intense matter, the matrix of intensity, intensity = 0; but there is nothing negative about that zero, there are no negative or opposite intensities. Matter equals energy. Production of the real as an intensive magnitude starting at zero. That is why we treat the BwO as the full egg before the extension of the organism and the organization of the organs, before the formation of the strata; as the intense egg defined by axes and vectors, gradients and thresholds, by dynamic tendencies involving energy transformation and kinematic movements involving group displacement, by migrations: all independent of *accessory forms* because the organs appear and functions here only as pure intensities. The organ changes when it crosses a threshold, when it changes gradient. (p. 153)

The BwO foster *intensities* that help to form the undifferentiated organism. Therefore, a BwO will be constructed in such as way so that *intensities must flow* and direct the becoming of the body (rather than sense). Deleuze and Guattari (1980/1987) provide examples of BwOs, such as the *masochist body*. Elaborating, Adkins (2015) writes:

> The masochist seeks to have only pain circulate, but he cannot convert his body entirely to intensities. He fights against the way it's organized, but he can only go so far. The limit that the masochist approaches (having only the intensity of pain circulate) but never reaches is the body without organs. Notice, though, that the body without organs is created by the masochist's experiment. It doesn't pre-exist the experiment, but it is generated simultaneously with the experiment as its limit. (p. 100)

Here, pain is the intensity that directs the body toward its BwO that it has constructed. The BwO does not exist in a literal sense but is rather the limit that the body chooses to tend toward. Intensities, rather than sense, guide the body's becoming toward this limit. Deleuze and Guattari (1980/1987) further discuss other BwOs such as the *schizo body*, the *drugged body*, the *hypochondriac body*, and the *paranoid body*. These bodies have the potential to be cancerous or empty, resulting in an end to the flow of intensities. How does one avoid this end? Deleuze and Guattari (1980/1987) state that:

> You have to keep enough of the organism for it to reform each dawn; and you have to keep small supplies of significance and subjectification, if only to turn them against their own systems when the circumstances demand it, when things, persons, even situations, force you to; and you have to keep small rations of subjectivity in sufficient quantity to enable you to respond to the dominant reality. Mimic the strata. You don't reach the BwO, and its plane of consistency, by wildly destratifying. That is why we encountered the paradox of those emptied and dreary bodies at the very beginning: *they had emptied themselves of their organs* instead of looking for the point at which they could patiently and momentarily dismantle the organization of the organs we call the organism. (pp. 160–161)

Note that this should sound familiar to D&G's idea of the plane of immanence in that there is always an active process of balancing, while not compromising, the infinite void of thought with a consistency that keeps the body from disintegrating into emptiness. The BwO, then, is a process in which the body joins the plane of immanence. Continuing, Deleuze and Guattari (1980/1987) write:

> How can we fabricate a BwO for ourselves without its being the cancerous BwO of a fascist inside us, or the empty BwO of a drug addict, paranoiac, or hypochondriac? How can we tell the three bodies apart? Artaud was constantly grappling with this problem. The extraordinary composition of *To Be Done with the Judgment of God*: he begins by cursing the cancerous body of America, the body of war and money; he denounces the strata….Artaud was constantly grappling with all of that, and flowed with it. (p. 163)

Deleuze and Guattari's (1987) approach to building a full, healthy BwO is to avoid organs that block intensities and deterritorialize too quickly. Fascism, for example, may produce an initial flow but leads to a cancerous end. Mathematics majors who study proof may begin with a healthy flow of intensities, but often, these flows stop and the BwO empties. Eventually, the student's study of proof merely amounts to "getting by" and understanding just enough to receive a minimally satisfactory grade. The flow of intensities ends and so does the BwO. In what ways does the genesis of proof allow for the construction of a healthy, active BwO while also avoiding, as part of this construction, "the cancerous body of America, the body of war and money" (Deleuze and Guattari, 1980/1987, p. 163)? In order to explore this, it is necessary, then, to make the move that D&G made from *The Logic of Sense* (1969) to *Anti-Oedipus* (1972). That is, mathematics and mathematics education must do more than simply explore the sense of a mathematical proof. Rather, we must also explore the *body* of mathematical proof as a means for producing the proof event and reaching the plane of immanence. It is only when the plane of immanence is reached that proof can be conceived as a becoming rather than as mode of advancement that ultimately serve the interests of the state and of capitalism.

Might the literature on embodied cognition be relevant to the BwO in terms of a positive modeling of proof as becoming? Consider, for example, Pier et al. (2019) in which the authors

demonstrate "that both dynamic gestures and logical statements are significantly associated with valid mathematical reasoning; however, based on these data, we cannot make causal claims about these relationships" (p. 54). Though the authors do not show causality, observe that the focus here appears to be on how bodily gestures (often involving how students model mathematical objects and processes with their hands) signify correct mathematical thinking. Though this work is representative of the literature on gesture and actions (see Alibali & Nathan, 2012; Nathan et al., 2014), the ultimate concern here is supporting cognition which, as I already discussed, is often in close proximity to transcendence and platonism. Consider de Freitas and Sinclair's (2012) critique, in which they, citing Deleuze and Guattari (1980/1987), indicate that

> One major concern with theories of embodiment is that they tend to locate knowing in the individual body and do not adequately address the collective social body, which is a material network-body connected and constituted through a rhizomatic lattice of material/social interaction. (p. 136)

It seems that what the authors are suggesting is that much of the embodied cognition research in mathematics education imply that mathematical thought is ultimately immanent to the subject/body, which places the bodily subject as transcendent. However, as I argued in the previous section (and what de Freitas and Sinclair appear to be suggesting), mathematical thought is not immanent to the body but rather the other way around: the body is immanent to thought (thought being the plane of immanence). Placing the plane of immanence (or the BwO) above the body/subject ensures that the body/subject never attains transcendence, as it will always be tamed and governed by the plane of immanence to which it is always connected to and never out of reach (and therefore not transcendent). Arguing similarly, de Freitas and Sinclair (2013) assert that

> The work on embodiment in mathematics education, however, has yet to adequately treat the materiality of mathematical concepts. In other words, while scholars have begun to attend carefully to the "sensuous" and the "corporeal" of the human body (in its everyday sense), there has been little to no interrogation of what it is to be a 'body' and how mathematics itself partakes of a body. In studying the learner of mathematics, much of the work on embodiment tends to fix the body in simplistic terms and to crystallize the discipline or "content" into a passive role. (p. 454)

These statements suggest that the research in embodied cognition is undertheorized with regards to materiality. However, the primary concern of the present research is to theorize about the materiality of proof and to understand how the BwO and the plane of immanence foster the sorts of intensities or becomings that lead to the proof events that produce mathematical knowledge as well as an ethic that has the potential to deterritorialize proof as a tool for capitalism. In the next section, I elaborate further on the dynamic genesis of proof in the context of the notion of *problematics*.

Proof and the Problematics of the Body of Proof

The dynamic genesis seems to have a relationship to what Deleuze calls problematics which, according to Smith (2006b), is a form of mathematical activity in which "a deduction moves from the problem to the ideal accidents and events that condition the problem and form the cases that resolve it" (p. 145). This is in contrast to the axiomatic approach where "a deduction moves from axioms to the theorems that are derived from it" (Smith, 2006b, p. 145). According to Brady (2020), problematics involve situations that resemble more informal thought experiments such as the Königsberg Bridges problem where one is presented with a set of islands and bridges that connect them: "is it possible to cross every bridge on a single path once and only

once?" (Brady, 2020, para. 12).This scenario corresponds exactly to a problem in graph theory so that a better understanding of the Königsberg Bridges problem allows one to better understand what conditions the underlying graph theory scenario. Consider another example from Smith (2006b) in which the author, referencing Deleuze, discusses the unsolvability of the quintic polynomial as an example of the problematic approach:

> In 1824, Abel proved the startling result that the quintic was in fact *unsolvable*, but the method he used was as important as the result: Able recognized that there was a pattern to the solutions of the first four cases, and that it was this pattern that held the key to understanding the recalcitrance of the fifth. Abel showed that the question of 'solvability' had to be determined internally by the *intrinsic* conditions of the problem itself, which then progressively specifies its own 'fields' of solvability. (p. 160)

Problematics, then, focuses not on solution and "solvability," but rather on *exploring* the conditions that *produce* the problem. While these conditions may indeed help to solve the problem, the solution itself is not the main focus. The absence of *produced* proof events, via the static genesis, places us closer to the world of axiomatics which merely stresses direct movements from stable premises to stable conclusions. Problematics makes primary the production of the conditions and events associated with a problem while axiomatics makes primary solutions and the more direct ways of attaining them.

For example, consider the statement *the sum of two even integers is equal to an even integer*. One can prove this in a straightforward manner using the following argument:

- Let x and y be two arbitrary even integers (where "even integer" is defined as an integer that is a multiple of 2).
- So, by definition of even integer, $x = 2n$ and $y = 2m$ where n and m are arbitrary integers.

- Therefore, $x + y = 2n + 2m = 2(n + m)$
- Since $x + y = 2(n + m)$ is also, by definition, an even integer, I am done.

Such algebraic arguments merely engage in "symbol pushing" where results are algorithmically deduced with little need to engage in the sort of explorations found in problematics. Geometric proofs in school mathematics involve "t tables," where statements are merely correlated with their warrants, and are also indicative of the axiomatic method. Regarding the number theory problem just discussed, one way to engage more with the problematics of this particular problem might be to work with pictures involving two groups of objects (with each of the two groupings consisting of some even number of such objects) as shown in Stylianou et al. (2015). With enough experimentation, one should eventually produce the idea or sense that, no matter how many objects constitute each grouping, as long as this number is even, it should be possible to combine the two groups such that the resulting, singular aggregate group can be partitioned into pairs of objects without any leftover objects. Understanding this condition, then, should suggest the role of a common factor of 2. It is this graphical approach to the problem that constitutes a problematics of proof that resembles a kind of experimentation rather than a purely algebraic, axiomatic formulation.

While Smith (2006b) seems to place an emphasis on problematics, it is worth noting, however, that he suggests both axiomatics and problematics are important to mathematics by quoting D&G:

> What we have, rather, are two formally different conceptions of science, and, ontologically, a single field of interaction in which royal science continually

> appropriates the contents of vague or nomad science while nomad science continually cuts the contents of royal science loose. (Deleuze and Guattari, 1980/1987, p. 367)

Here, "nomad" science would be akin to the problematic mode while "royal" science would be akin to axiomatics. Nonetheless, Smith (2006b) states that "In Deleuzian terms, one might say that while 'progress' can be made at the level of theorematics and axiomatics, all 'becoming' occurs at the level of problematics" (p. 158). This is an important idea in the production of proof-sense and the heuristic approaches to proof given in the literature. While key ideas, proof images, and proof schemes are themselves closer to that of the static genesis of sense, their production is the result of the dynamic genesis. In the case of the proof discussed earlier in this section, the arbitrary experimentation with the objects followed by the realization of the notion that there will always be an even number of objects constitutes the dynamic genesis of proof sense while the realization of the notion itself, without its production, is closer to that of the static genesis. The dynamic genesis of proof demonstrates that proof, ultimately, does not emerge from stable axioms or cognitive conditioning but rather from the intensities and asignifications of the body. These becomings, then, produce a proof sense that serves as a ground for the events that actualize mathematics, organizing thought into states of affairs which are then given a name or attribute via a theorem statement. While I referred to proof-sense as a "ground," its dynamic genesis also constitutes a kind of "groundlessness" (Deleuze, 1969/1990) with the capacity to engage with the immanence of the body which ultimately grants us access to *THE* plane of immanence.

Research Approach

In this work, I utilize Jackson and Mazzei's (2012) "thinking with theory" approach to understanding data. This approach involves using concepts or "schematic cues" from Deleuze

such as becoming, difference, desire, deterritorialization, and intensities to think and write through qualitative data. For example, Jackson and Mazzei (2012) describe how desire can be used to study the anecdotes of "first-generation women faculty and administrators in order to understand their educational, socio-cultural, and professional experiences" (p. x), Using a modified notion of Deleuzian desire, what the authors refer to as "desiring silence," Jackson and Mazzei (2012) analyze moments of silence in relation to an anecdote with Cassandra, a Black faculty member, and her meeting with a white administrator. When taking this Deleuzian approach, the authors also indicate that production is important. What are the silences and desires producing? In this case, it is protecting the privileged positions of white university administration. For my study, I take up a similar approach to using Deleuze's ideas in analyzing proof as a kind of desire (or pure intensity or difference) and understanding what productions and assemblages result in the moments of "becoming proof" as described by Neel (2019), in a story he wrote about his own experiences with a particular problem connected to his dissertation work.

In analyzing this story, I also draw upon Stinson and Bullock's (2012) work on *critical postmodern theory* (CPT), a variation of critical theory (CT), which the authors describe as "a hybrid theory that offers a *praxis of uncertainty*" (p. 41). Drawing on Freire (1970/2000), Stinson and Bullock (2012) describe such a praxis "as a continuous cycle of action and reflection in which sacrificing action equates to empty verbalism while sacrificing reflection equates to mere activism" (p. 49). Citing Pais, Stentoft and Valero (2010), Stinson and Bullock (2012) state that "this moment also has the potential to move researchers away from an agenda that primarily explores questions of *how* to improve mathematics teaching and learning toward an agenda strongly concerned with the often forbidden question of *why* mathematics education (Pais,

Stentoft & Valero, 2010)" (p. 45). By utilizing CPT as described by Stinson and Bullock (2012), my aim is to show how proof itself, as a plane of immanence, demands "a reconceptualization of the very structure and existence of mathematics" (p. 51).

In the next section, I focus on the story written by Dr. David Neel from the book *Living Proof: Stories of Resilience Along the Mathematical Journey* (2019) published by the American Mathematical Society. This story is an autobiographical piece describing a proof experience near the end of his doctoral studies that involved his dissertation supervisor Ken. As Neel (2019) elaborates, he had accepted a faculty position at a university but had not yet finished his dissertation. I use many of D&G's ideas described previously as a frame for viewing and analyzing his story.

Becoming Proof and Reaching the Plane of Immanence

So how does the body of proof produce the events that allow for a problematic approach to mathematics and in what ways do these events contribute toward proof as an unstable becoming on the plane of immanence? In the beginning of Neel's (2019) story, he writes "I'd found a shorter proof for the main result. We like short proofs. But when they get too short... 'Yes, it doesn't quite seem like the main result of a thesis anymore does it?' Ken said" (p. 84). David had not yet reached the plane of immanence from which his dissertation would emerge.

David and Ken decided to meet at Ken's winter home to work on a more challenging problem that would finish his work and allow him to graduate. However, as they engage with the problem, they are met with further difficulty and are unable to "breakthrough." Neel (2019) writes:

> But by three days in, though we knew the problem more intimately, the long sessions at the whiteboard in his basement office had failed to produce the breakthrough. I felt

> claustrophobic, anxiety-ridden at the closeness. Each day, all day, we were together.
>
> Were we stuck? (p. 85)

Both David and his supervisor Ken are still within a more traditional environment of an office complete with a whiteboard. So far, what has occurred appears more akin to a static genesis where the body appears to play little to no role. They are, perhaps, seeking a solution but the material production of that solution is not important, at least not yet. It is at this point that both Dr. Neel and Ken decide to leave the interiority of the "basement office," and perhaps the subjectivity of the mathematician, to hike in a nearby state park where they continued to try and work out the details of the final results. Notice how Neel (2019) describes the hike:

> The hike to lunch took closer to 45 minutes, but the weather sparkled and my flat feet didn't feel too bad. Most importantly, the shifting angles of the light, the mixed and faded shades of evergreen trees and grasses, the brown of mud and trail, the fact that we were watching our feet, and the sky, and the undergrowth, this all somehow focused our concentration into a pair of tiny sparks, dancing amongst dry tinder. (p. 85)

I argue that the "shifting angles of the light, the mixed and faded shades of evergreen trees and grasses, the brown of mud and trail, the fact that we were watching our feet, and the sky, and the undergrowth" is indication that Neel is experiencing a different kind of process that is unlike the cognitive heuristics and strategies that can be found in the mathematics education literature on proof. This process is a construction but not of an axiomatic proof. Rather, it is the building of a body without organs that facilitates the intensities necessary to engage with proof as a problematic process (rather than an axiomatic process) and producing the necessary events for the eventual actualization of a rigorous proof. Neel (2019) continues: "I almost worried when we

stopped in the clearing. We had no new insight yet, but something loomed, felt nearer somehow. We had stated the problem with more clarity, or with a more promising perspective" (p. 85). Recall that axiomatics "moves from axioms to the theorems that are derived from it" (Smith, 2006a, p. 145) while problematics involves "moves from the problem to the ideal accidents and events that condition it and form the cases of solution that resolve it" (Smith, 2006a, p. 145). I argue that David and Ken's work is closer to the latter, that of problematics, since there appears to be a move from the problem to concerns about problem clarity and the formation of "a more promising perspective." The two are discovering and working with the conditions of the problem itself rather than a linear move from the problem's assumptions to the resulting theorem. It is also interesting to note the role that Neel's (2019) dissertation supervisor has in directing the intensities of this proof:

> "You know,'' Ken said, repacking his pack, "we could just go back, but it seems like we're on the right track. Do you want to hike in a bit deeper and keep talking?'' I ignored my aching fallen arches and nodded. Follow Ken's lead. He's the one who's been through this all before. (p. 85)

Perhaps this suggests that the role of the proof instructor is not only to help the student gain mathematical understanding but also to encourage the flow of material intensities toward the plane of immanence and the event that actualizes a proof. In Neel's (2019) case, the intensities that are transmitting involve a kind of physicality with the environment:

> We reached another split in the path. We may have made a joke about graph theory. Two paths, but the shorter one was marked: "Path Closed for Maintenance." Still two options, really: forward or back. We do not have a map. Ken is pretty sure he remembers where this forward path emerges. It should be fine. Keep walking. Keep talking about

> combinatorics.…It feels as if this has become some elemental struggle. And we're still, somehow, talking about combinatorics. (Neel, 2019, p. 86)

Many different components are entering into an assemblage here: splits in paths, jokes about graph theory, the absence of a map, and combinatorics all functioning with the proof that is being worked out. The "elemental struggle" that David speaks of is the assemblage components all entering into differential relation with each other. Difference is now taking priority over the stable axiomatic approach. The BwO that is being constructed here encourages and facilitates a sense of intensity, as if David and Ken are not just engaging in constructivist mental activity but also *getting closer to something*. This sense of nearness, however, is not so much related to cognition as it is related to that of the body. David and Ken are entering into proximity of the proof event which is produced by the intensities associated with the BwO that the two are constructing. Does this BwO somehow reside within David and Ken? I would suggest, drawing on Philp Goodchild's idea of being "within" thought (Timeline Theological Videos, 2014), that David and Ken are a part of it but that this BwO does not exist within the subjects of David and Ken. Rather, David and Ken reside within it (the BwO). When they make their move to hike in the nearby state park, their bodies become a part of a vast assemblage consisting of the Earth and the hike as well as the mathematical elements they are pursuing. This assemblage is a deterritorilization process that returns proof to an unstable, unconditioned, primordial state, away from the territorialized white board and office space they had previously been working in. This previous space ultimately emphasizes the transcendent and sensing subject (the mathematician), a static genesis, of which proof resides and originates within. However, this centering of the subject and the axiomatic method can only go so far. An axiomatic approach to this problem does not produce a process that is "worthy of the event" (Deleuze & Guattari, 1991/1994, p.

160). For proofs such as the one that David and Ken worked on, one must deterritorialize, and a collapse of sense must occur, a collapse into the body and toward the plane of immanence. This collapse gives rise to the BwO that facilitates proof at the level of the virtual, guiding it with material intensities (rather than mental processes) toward the actualization of the proof event. Neel (2019) writes:

> But now, we have it. It snapped into place. Now, we are sketching it out more fully, outlining it for each other, repeating, so that if only at least one of us can somehow walk or crawl free once more into that world outside this state park we can explain it to someone. (p. 86)

This is the moment when the proof event actualizes. However, even after this moment, David and Ken are still within proximity to it as it has only just occurred. The "snapping into place" is the moment of the event when the solution is "stumbled into" rather than axiomatically deduced. Observe that even after this proof event passes, David and Ken are still "sketching it out more fully, outlining it for each other, repeating." The event has passed but the actualization process continues. Neel (2019) indicates a need to "explain it to someone" which suggests something akin to a validation process in which a community deems a proof to be logically correct and rigorous, thereby establishing the proof in the actual.

At the end of the story, Neel (2019) looks back on the experience not by discussing the proof's mathematical surplus value or its logical correctness, but by reflecting on a kind of ethic:

> The moral is clear: care for one another, keep walking, do not despair. One other moral: Ken's example, his kindness and generosity. He was a model mentor and a

good man. I had hoped and expected more years and many more chances to thank him. (p. 86)

In the context of this particular proof experience, this sentiment seems to reflect what Roquet (2014) refers to as *cosmic subjectivity* which is “a form of self-understanding drawn not through social frames, but by reflecting the self against the backdrop of the larger galaxy” (p. 124). While a literal galaxy may not play a role in Neel’s (2019) experience, it is *THE* plane of immanence, perhaps, that functions as this universe, this “opening up of possibilities” (Rota, 1997, p. 191), with the capability of producing events that generate new mathematical results while also fostering dispositions within the mathematician that are not typically associated with proof. This is the power that mathematical proof has as a becoming of the asignifying body rather than as a signifying, axiomatic strategy performed in the name of progress that ultimately serves “certain (technoscientific) ends within twentieth-century capitalism” (Rotman, 2000, p. 36). These ends constitute “the cancerous body of America, the body of war and money” (Deleuze and Guattari, 1981/1987, p. 163). They are what axiomatic proof can often lead to. The asignifying production of proof allows for mathematical progress to be made while also resisting this neoliberal posture that often accompanies scientific work. In Neel’s (2019) case, this resistance can be found in the “moral” that he concludes his story with.

Like Artaud, as referenced in Deleuze and Guattari (1980/1987), Dr. Neel found a way to flow with the proof assemblage that he and his supervisor constructed. To flow in this manner is to deterritorialize proof itself so that what is left is a primordial, cosmic body that is perhaps, by some mysterious process, untouched by the effects of axiomatics and capitalism itself. This deterritorialization positions mathematical thought in the material

world, on a single plane of immanence, *THE* plane of immanence. It allows proof to take on a form that forces the mathematician to see the traditional elements of proof "as minor transformations within this larger and more persistent frame" (Roquet, 2014, p. 147). Roquet (2014) writes that "this reframing serves an ethical purpose, downplaying self-interested social desires and emphasizing instead how the self emerges from the profoundly impersonal forces at work in the larger universe" (p. 158). These "forces" constitute the assemblage that Neel (2019) and his supervisor Ken construct during the "proof-hike" activity they engage in. In reaching *THE* plane of immanence, Dr. Neel was able to see proof as having this ethical dimension, one that avoids capitalist self-interest and instead partakes in a becoming that allows for these new possibilities (both mathematical and otherwise) to emerge in the first place.

Implications and Conclusions

Several implications regarding proof, the teaching and learning of proof, and research methodology emerged from my analysis of Dr. Neel's story. First, my analysis suggests that mathematical proof, at its unconditioned genesis, is constituted by material intensities (or flows) that do not resemble anything static or stable. It is not simply static key ideas and heuristics that constitute proof but rather their virtual production. Proof is not a purely cognitive activity. This can be seen when Dr. Neel and his supervisor Ken decide to leave the basement office in which they had been working. The traditional environment of the mathematician is replaced with the hike, which provides the environment for which intensities can proliferate. For example, Neel (2019) writes: "we were watching our feet, and the sky, and the undergrowth, this all somehow focused our concentration into a pair of tiny sparks, dancing amongst dry tinder" (p. 85). In this story, the hike is the BwO that both Dr. Neel and Ken are constructing. This BwO is the body or

assemblage of proof, rather than merely the body of a mathematician/subject, which fosters the intensity or concentration: the "tiny sparks, dancing amongst dry tinder" (Neel, 2019, p. 85). While there are clearly mental processes at play, they are not processes regarding, as de Freitas (2013b) writes, "recognition" and "(mis)recognition" (p. 592) of a solution. Instead, these processes are intensities that find themselves among other such forces and flows such as walking, elemental struggles, stating the problem differently, deciding whether to go forward or backward. These are what make proof "worthy of the event" (Deleuze & Guattari, 1991/1994, p. 160), that is, these are what *produce* proof as an event in dynamic genesis as opposed to simply actualizing it in a static one. The BwO, as the entity that produces, via intensities, these proof-events, is even more fundamental than any proof heuristic or completed proof argument. It is not an individual nor their cognition, nor the heuristics they employ that actualizes a proof or the possibility for a proof. Rather, it is the proof assemblage or BwO that actualizes such products.

By making primary the body of proof (or the BwO of proof), a realization emerges: proof, at its genesis, is not grounded in the subject (the mathematician, the student), nor the social relations that exist between subjects, nor in any stable, ideal state of affairs just waiting to be discovered. Rather, mathematical proof is founded on the groundless consistency of the asignifying body or plane of immanence. Since the plane of immanence is a material plane, it is never completely out of reach. We may not be able to comprehend it, but we can be connected to it as demonstrated in Neel's (2019) story. Furthermore, this plane resists the transcendent status traditionally associated with proof and mathematics, replacing transcendence with immanence. By making this replacement, proof is no longer a stable identity defined and conditioned by the mental processes of the subject or subjects but rather "an opening up of possibilities" (Rota, 1997, p. 191), whether these be mathematical, ethical, or otherwise. This opening up is consistent

with Stinson and Bullock's (2012) "praxis of uncertainty" when, in referencing Freire (1970/2000), "*speaking a true word* and *transforming the world* are both left open to multiplicitous possibilities." (p. 49) and where "change (i.e., deconstruction/reconstruction) must be forever in uncertain and somewhat directionless rhizomatic motion" (p. 49).

This, of course, has implications for the teaching and learning of proof. For example, the immanent view of proof presented in this paper is not entirely inconsistent with notions of constructivist-inspired active learning approaches (see Freeman et al., 2014). After all, when a BwO is constructed during the assemblage of a proof, a student's sense making becomes part of the intensities that are fostered by such a BwO. This, perhaps, suggests that teaching involves actively engaging students via processes that encourage exploration, curiosity, and wonder rather than simply solutions to problems. With that said, more must be taken into account here since the BwO is primarily a material and not a cognitive construction. The construction of a proof BwO involves the *integration* of material forces (including mental processes) that flow rhizomatically along various mathematical, cognitive, ethical, physical, poetic, and "cosmic" lines. By seeing these dimensions of proof as a rhizomatic integration or assemblage, even aspects such as problem-solving and mental processing *become bodily intensities first rather than activity that is contained within the sensing transcendent subject*. However, the active learning tradition usual does not see constructivism in this light, and it is in this sense that active learning does not partake in the immanent view. While the embodied cognition literature (e.g., Pier et al., 2019), a kind of subset of the active learning tradition, takes into account the physical human/subject body in the learning of proof, it too faces insufficient theorization in relation to materiality as given in the critique from de Freitas and Sinclair's (2013). This critique suggests that even the body can become a simple state of affairs that is capable of merely signifying mathematical

meaning. Therefore, it is not just the connection to one's literal, physical body in relation to cognition that fosters the kind of BwO that is located at the dynamic genesis of proof itself. The hike that Neel (2019) and his supervisor take is an attempt to initiate an external encounter with the proof event. This is an important aspect to the immanent view of the learning of mathematical proof. Using the language from de Freitas (2013b) concerning the event: proofs "occur rather than exist" (p. 587) and, therefore, the learning of mathematical proof becomes a question of becoming rather than that of a mental construction designed to imitate or approximate a particular mathematical ideal. Therefore, we should trouble the traditional lecture approach to the teaching and learning of proof as such methods are closer to the static genesis rather than a dynamic one. Traditional lectures do not allow for becoming to occur as easily and we should find ways to subvert this approach so the students can more actively engage in the construction of asignfiying assemblages and toward the plane of immanence.

Neel's (2019) story suggests that the sort of constructivism necessary for becoming-proof is not located exclusively within the sensing subject or amidst the interplay of subjects but rather in the assemblages which the subject is a part. How, then, do we approach the teaching and learning of proof? When Neel (2019) and his supervisor Ken decide to hike in a nearby park, Ken suggests the possibility to continue on the trail (and perhaps the line of thought that had opened up) or to turn back. Ken knows that his role as an instructor is to encourage the intensities that produce the necessary proof events. By relinquishing control over the situation and allowing his student to decide where to go next, Ken removes himself as a potential block of the intensities that are producing the event that actualizes the proof (the snapping into place). This is much easier to do when Ken himself has never proved this particular result. Therefore, Ken's role is not to leave a logical, axiomatic "bread trail" toward the event (this would be more

akin to the adequation of a proof argument to its corresponding theorem statement, that is the static genesis) but rather to facilitate the material intensities that permeate the proof event. Even Neel (2019) seems to recognize this when he writes "He's the one who's been through this all before" (p. 85). This statement acknowledges Ken's ability to reckon with proof intensities and events rather than simply knowing the answers to problems and the specific cognitive moves to get those answers. One might read Neel's (2019) statement as "He's the one who's reached the plane of immanence before." Ken's ability to facilitate the flow of intensities rather than simply directing cognitive processes is consistent with this immanent view.

In this paper, I have argued that the proof process in Neel's (2019) story is not the centering of transcendent, signifying logical and mental processes of subjects (or between subjects) nor is it a matter of a platonic imitation of an ideal mathematical world. Neel's (2019) story suggests that ultimately, proof begins with the groundlessness of bodily materiality. St. Pierre (2020) writes that "in an ontology of immanence, there is no ground, no foundation, no beginning, no origin anywhere. There never has been. The world does not and cannot contain separate, stable entities. Everything is always in continuous variation, becoming" (p. 486). Proof, with its dynamic genesis on the plane of immanence, is a matter of feeling, of knowing when to move forward or backward, and a matter of removing or avoiding potential blockages of these intensive movements. It is, also, a question of ethics. The "snapping into place" that Neel (2019) describes is intrinsically connected to his cosmic subjectivity, an "opening up of possibilities" (Rota, 1997, p. 191). Proof, then, from an immanent view, is a process for understanding oneself in the context of this groundless plane. The mathematician who traverses this plane is not one who determines mathematical truth, uncovers it, and decides what one "ought" to do mathematically. It is not merely a process for extracting a mathematical surplus value via a static

genesis. Rather, proof is about reveling in the uncertainties and possibilities that the plane of immanence has to offer. Proof is, in this sense, antiauthoritarian and interferes with a capitalist view of proof which situates mathematics as an instrument of logical and technological advancement. If we agree with the immanent perspective, then our motivation and approach for teaching students to prove mathematical results should be consistent with this view.

What the Philosophy of Gilles Deleuze and Félix Guattari has to say about the Teaching and Learning of Proof

In this paper, I use ideas from post-structural philosophers Gilles Deleuze and Félix Guttari (or D&G), as a framework for understanding the teaching and learning of mathematical proof at the undergraduate level. Additionally, I draw upon a story from the 2019 book *Living Proof: Stories Along the Mathematical Journey*, published by the American Mathematical Society (AMS), as a way to further make sense of the teaching and learning of proof as a *becoming* rather than as a *platonist ideal*, that is, as something that is a process of a subjective or intersubjective mental construction or as a means to reach some sort of objective resolution. When viewing mathematical proof through the lens of Gilles Deleuze's 1969 text *The Logic of Sense*, we find that proof as a transcendent form is ultimately a static one that is disconnected from the material, asignifying body and, therefore, without a true genesis. Proof, or the genesis of proof, does not reside in the lines of text that constitute its presentation (the formalist perspective), nor does it exist in any objective state of affairs (the platonist perspective), nor does it reside in subjective sense or meaning (the intuitionist perspective). These three perspectives, drawn from Rotman (2000), are transcendent and only serve to essentialize the "nature" of proof. Rather, D&G would argue, the true genesis of proof resides in *immanence* or the dynamic material forces, both physical and metaphysical, that constitute our more immediate reality.

In the first half of this paper, I discuss the literature surrounding proof, mostly at the undergraduate level, and interpret this literature with D&G's concepts to argue that proof, in both the mathematics education space as well as the space of the professional mathematician, is often situated as a static, transcendent form that lacks a genesis in immanence. Then, I use D&G's ideas along with a story from *Living Proof* to argue that mathematical argumentation also

involves the reversal of this transcendent thinking, that is, as something grounded in a dynamic becoming first, prior to its actualization in mathematical text, facts, or meanings. In the second half of this paper, I discuss the implications for the teaching and learning of mathematical proof at the undergraduate level and touch on some implications for research on proof.

Proof as Transcendent, Static Form

In Deleuze's (1969) book *The Logic of Sense*, the author asks the question: where does meaning reside? Is it located in the propositions of language itself or in the states of affairs that these propositions actually refer to? Smith (2022) elaborates on an example from Deleuze (1969) regarding the proposition "The Battle of Waterloo" which refers to a historical state of affairs, that is, something that actually occurred. Where does the battle exist? It seems strange that it would reside in the proposition "The Battle of Waterloo" itself as this is just a sequence of words. Additionally, it seems equally strange to locate the "battle" in its actual state of affairs since this is merely matter mixing with matter (bodies intermingling). Therefore, the "battle" cannot be located in its states of affairs either. Smith (2022) writes that:

> We can attribute 'Battle of Waterloo', for instance, to a particular state of affairs, but what we find in that state of affairs are bodies mixing with one another: spears stabbing flesh, bullets flying through the air, cannons firing, bodies being ripped apart. Strictly speaking, the battle itself exists nowhere except in the expression of my proposition, which attributes 'Battle of Waterloo' to this mixture of bodies. More precisely, we could say that the battle itself merely 'insists' or 'subsists' in the proposition. Hence one of the fundamental theses of *Logic of Sense*: sense is to propositions what attributes like 'Battle of Waterloo' are to states of affairs. They are pure events that subsist or insist in both propositions and states of affairs. (p. 8)

Arguing from Deleuze (1969/1990), Smith (2022) concludes, then, that the "battle" exists in the *expression* of the proposition as well as in the *attribute* of the state of affairs. Both *expression* and *attribute* corresponds to *sense* or *meaning* which Deleuze (1969/1990) describes as a static, sterile, and incorporeal *event* or *effect*. By taking into account the expression of the proposition and the attribute of the state of affairs, the two are adequated and the proposition "The Battle of Waterloo" now has a clear referent. The same can be said of proofs and theorems. A written proof is akin to a set of mathematical propositions and a theorem is its corresponding state of affairs. When I write a proof successfully, the expression associated with the written proof connects with the attribute of the theorem (a state of affairs) and the two become adequated. Proof, whether in the reading or writing, requires a complicated interplay between expression and attribute that form mathematical meaning. It is this meaning that is primary, rather than the mathematical propositions and states of affairs themselves. If I did not have mathematical sense first, then there would be no way to understand proof in a formal manner.

These three domains, propositions, states of affairs, and the sense that is associated with them can be imported into the world of mathematical proof where "propositions" correspond to something like the text of a mathematical proof while the corresponding states of affairs could be the actual "facts" or "truths" that are discovered and imitated by the mathematician in the theorem-proving process. Finally, "sense" would correspond to the underlying mathematical meanings and heuristics that correspond or result from an already completed, written proof or that are involved with the production of a proof. These three domains seem to be similar to the formalist, platonist, and intuitionist views of mathematics as described in Rotman (2000). These perspectives attempt to define mathematics in a particular way by foregrounding a worldview about mathematics. The formalist sees the work of the mathematician as merely applying

abstract, value-free logical rules to arbitrary symbols or propositions, the platonist sees mathematics as accessing or discovering "truth" and mathematical "facts," while the intuitionist choses to see mathematics as, ultimately, a product of the mind and of human meaning-making. All three of these domains are transcendent in that they are ultimately disconnected from material existence. Additionally, all three domains are, in some capacity, *signifying* in that they involve the mimicking, approximation, and imitation of mathematical ideals or sense. Latent in the field of mathematics, then, is this propensity to reproduce a standard. If this reproduction fails, then it is seen as deficient, incorrect, or, as Deleuze and Guattari (1987) might suggest, "deviant."

Much of the mathematics education literature on proof, I would argue, seems to align itself with aspects of the formalist, platonic, and, intuitionist perspectives. For example, in Weber (2001), the author utilizes an *information processing* framework to evaluate undergraduate and graduate students' strategic knowledge (or lack thereof) "or heuristic guidelines that they can use to recall actions that are likely to be useful or to choose which action to apply among several alternatives" (p. 111). This research approach seems to emphasize individual knowledge and heuristics. The information-processing aspect may even suggest a degree of formalism involved with student proof processes. The research on proof schemes (e.g., Harel & Sowder, 1998, 2003; Recio & Godino, 2001; Housman & Porter, 2003) focuses on students' frameworks for approaching proof tasks such as the use of a limited number of cases to draw general conclusions, appeals to authority and intuition, etc. These frameworks, or schemes, emphasize the mind as the ultimate authority in the proof process and, therefore, have some connection to the intuitionist perspective of mathematics. In a case study by Weber (2004) regarding the teaching of an undergraduate introductory real analysis course, the instructor in the study

emphasized that an understanding of logic, solid procedural ability with symbolic manipulations, and intuitive understanding of mathematical meanings were necessary in the learning of advanced mathematics. As was the case with Weber (2001), the formalist and intuitionist perspectives seem close here.

While not technically education research, Brown (1997) does appear to adhere to elements of platonism when discussing the role of pictures as part of the proof process: "some 'pictures' are not really pictures, but rather are windows to Plato's heaven.…As telescopes help the unaided eye, so some diagrams are instruments (rather than representations) which help the unaided mind's eye" (p. 174). This suggests that proof is a method for imitating or mimicking objectively true "facts" about reality that are located in some kind of transcendent space. While this type of thinking does not seem to manifest itself explicitly in the mathematics education proof literature, I argue that the platonist view is often *implicit* in these studies by assuming notions related to "truth" and justifiable mathematical "facts." For example, Czocher and Weber's (2020) discuss proof as a "cluster category," a concept related to Wittgenstein's notion of *family resemblance*, which "has a collection of properties that an object can satisfy to 'count toward' category membership, but no single property is necessary or sufficient for category membership" (pp. 59–60). Some of these properties include that "proof is a *convincing justification* that will remove all doubt that a theorem is true for a knowledgeable mathematician" (p. 61) and that "A proof is a *perspicuous justification* that is comprehensible by a knowledgeable mathematician and provides the reader with an understanding of why a theorem is true." (p. 61). This emphasis on true theorems suggests the platonist perspective.

While there may be processes involved with these proof studies (cognitive, social, etc.), I want to suggest that these processes ultimately situate truth as an "*adequation* to states of affairs"

(Smith, 2022, p. 10) that, for Deleuze (1990), is sterile and static, failing to “be a matter of *production* within sense” (Smith, 2022, p. 10). What this means, for Deleuze (1990), is that by aligning with one or more of the formalist, platonist, or intuitionist perspectives, truth, and therefore proof itself at its genesis, is merely a game of referencing, naming, or signifying some ideal that, according to Rotman (2000), is likely a social construct to begin with. It is in this sense that all three of these worldviews are connected to platonism. This world of the signifying can also be seen in the above studies’ recommendations for the teaching and learning of proof. For example, in Czocher and Weber (2020), the authors suggest that instructors give appropriate credit to their students for their proof work, even if the result is not a fully rigorous, professional proof. After all, a proof can be convincing while still not being technically correct. It can also be correctly grounded in appropriate axioms and definitions with the argument itself still being flawed. By emphasizing students’ ability to correctly engage in certain proof “virtues,” as Czocher and Weber (2020) put it, there is an implicit assumption that these virtues are ideal and exist as timeless properties of the proof process. There is also a sense, with much of the above literature, that we must be changing students’ proof practices so that they resemble that of the expert mathematician and, therefore, proof instruction should emulate these aspects. The work on proof schemes (e.g., Harel and Sowder, 1998, 2003) appears to suggest that instruction should move students’ schemes from less mature ones (such as empirical and authoritarian schemes) toward more professional, analytical ones. From an instructional point of view, then, platonism manifests itself in mimicking that of the expert mathematician, particularly aspects that correspond to the static genesis of proof.

None of this is to say that this type of research is somehow misguided as they all present important aspects of the proof process. There is no doubt that proof engages in platonism and

this dimension should clearly not be ignored. However, I situate the proof literature in this manner in order to locate what has not been explored yet: proof as an unstable, material *becoming* that has an anti-platonist and asignifying genesis to it alongside its platonistic components. This is ultimately where D&G find themselves with regards to language and linguistics: there is no stable ground to language and sense inevitably will collapse into the world of the body. A world where nothing signifies or "means" anything and where only intensities and productions reside. In the next section, I describe this domain and its relevance to mathematical proof.

The Dynamic Genesis

As mentioned in the previous section, Deleuze (1969/1990) indicates that the static genesis does not take into account how sense and events are materially produced and, therefore, places limitations on how meaning and truth are understood. Under the static genesis, "truth," the "correct" adequation of a proposition to its state of affairs which occurs in mathematics during the proof process, is merely an ideal to imitate rather than something that is active and dynamic. Deleuze (1969/1990) introduces a modification to the static genesis by developing the idea of the *dynamic genesis* of sense which no longer simply places emphasis on meaning, or in the case of proof, as that which connects states of affairs (mathematical fact) with propositions (formal proof argument) but, rather, is concerned with how proof is produced in the first place. The upside to this is that there is more to truth rather than simply a kind of platonist ideal. Truth and meaning are now infinitely complex, irreducible, and much more interesting. The potential "downside" to this is that truth and meaning can never be pinned down and truly knowable. This is the position I take in this paper. That is, in addition to ruling out proof as primarily constituted by the lines of its text (proof as a proposition), or located in the mathematical "facts" that these lines refer to

(proof as a state of affairs), I also rule out proof as being in primary relation to its sense which still anchors proof to signification and its connection to its propositions and states of affairs. As D&G explore in both *Anti-Oedipus* and *A Thousand Plateaus*, there is something else that occurs prior, or perhaps alongside, propositions, states of affairs, and the sense associated with them. Rota (1997) also questions this theorem/proof binary, suggesting that what is more fundamental to mathematics is to see proof as "an opening up of possibilities" (p. 191). This opening seems to resemble Deleuze's (1994) idea of the plane of immanence or, as Spindler (2010) put it, "the horizon out from which thinking as such can take place, and thus constitutes the internal condition of thinking" (p. 151). I want to suggest, as part of Rota's (1997) discussion, that we also incorporate notions of bodily materiality as part of the possibility of proof as this is what D&G seem to do with regards to language. That is, language has its genesis in the asignifying body, or the body without organs (BwO). I discuss this notion next.

When D&G discuss the BwO, it is not necessarily in reference to a literal body. As will soon be discussed regarding embodied cognition in mathematics education, by placing too much emphasis on the literal, physical body, we merely emulate the thinking subject which situates us again in the world of sense and signification. While the literal body does have the capacity to engage in unorganized becoming (via medical procedures, athletic feats, etc.), it is best to speak about the BwO which emphasizes the way in which bodies (literal or otherwise) or assemblages reject the organization and stability that society often expects. In *The Logic of Sense*, Deleuze (1969/1990), discusses how sense often collapses into the depths of the body, citing artists such as Antonin Artaud, who often expressed coherence while gradually spiraling into noisy, but poetic, incoherence. However, this sort of collapse occurs in everyday situations as well such as debates that begin with polite discussion but end with rude name-calling and

nonsensical accusations, computer programs that crash or glitch, and moments of humorous absurdity and laughter. Rather than using the word sense here, since there is none in these situations, D&G utilize the term *assemblage* to denote the synthesis or integration of asignifying forces that have the potential to form something novel. Antonin Artaud's work may be incoherent but it is also decolonizing in that it serves the purpose of exposing harmful capitalist norms in society. A computer program may crash or produce glitches during use but this collapse of sense between a program's code and its output may also provide new ways to interact with the software. For example, in the gaming community, fans of "tool-assisted speedruns" often purposely overload the game software in order to achieve total control over the program for further creative purposes (see Lord Tom, 2016; "Tool-assisted speedrun," 2024). These unorganized bodies, or BwOs, are assemblages that form a becoming, a dynamic genesis, from which something new and more desirable is acquired. This bringing together of an assemblage is also rooted in D&G's ideas concerning *desire* as something that is not in response to some "lack," but rather as the proliferation of untamed difference in itself. This type of desire is an *intensity* rather than a signifying, Freudian response to some psychological issue. Intensities are affective, operate at the level of the BwO, critical in the formation of the subject, and constitutive of all becomings and of the dynamic genesis. In Deleuze and Guattari's (1987) *A Thousand Plateaus*, the authors discuss the example of an egg and the developing organism that resides within. Prior to the formation of the organism's organs, the egg is only constituted by intensities that slowly begin to shape and bring into being a body with the potential for subtle, physical variations. The egg, prior to this organization, is a body without organs. It's important to note that, for D&G, *the egg is not a metaphor* for the BwO. Yet, at the same time, it isn't the literal egg either but, rather, that which allows intensive forces to flow. These intensities are a

becoming. They are the organism's genesis. How does proof function like that of an egg and how can it be theorized as a BwO?

It may seem natural to assume that mathematics education research in embodied cognition may support the utilization of the body to understand the teaching and learning of mathematics. For example, Alibali and Nathan (2012) explore how gestures in instruction ground and aid understanding and learning of mathematical concepts. Similar results were found in Pier et al. (2019) concerning how gestures embody valid proof practices, indicating that "A stronger understanding of the verbal and discursive structure of valid mathematical arguments could lead to recommendations for how teachers might support students who struggle to express themselves mathematically" (p. 55). This work is certainly a worthwhile line of inquiry as it demonstrates that nuances that are entangled with valid proof approaches. However, as de Freitas and Sinclair (2012) state:

> One major concern with theories of embodiment is that they tend to locate knowing in the individual body and do not adequately address the collective social body, which is a material network-body connected and constituted through a rhizomatic lattice of material/social interaction (Deleuze & Guattari, 1987). (p. 136)

This critique can be applied to the embodied cognition literature cited above. By focusing on the literal, physical body rather than the intensive assemblages, mathematics is reduced to the thinking subject as the arbiter of mathematical meaning. This, as has been discussed, leads to platonic, static conceptions of proof. De Freitas and Sinclair (2013) also note that:

> The work on embodiment in mathematics education, however, has yet to adequately treat the materiality of mathematical concepts. In other words, while scholars have begun to attend carefully to the "sensuous" and the "corporeal" of the human body (in its everyday

> sense), there has been little to no interrogation of what it is to be a 'body' and how mathematics itself partakes of a body. (p. 454)

I tend to agree with de Freitas and Sinclair's (2013) suggestion concerning redirecting our focus on the literal, "corporeal" body to the body of mathematics and mathematical concepts themselves. The work of de Freitas and de Freitas and Sinclair are rare examples of seeing mathematics itself as an intensive becoming. For example, in de Freitas (2016), the author states, that "this is not a platonic philosophy of mathematics (in which number is an abstract ideal that transcends matter), but a philosophy of immanence, where *number is implicated in a vibrant and indeterminate matter*" (p. 4). I situate my approach to proof closer to de Freitas and de Freitas and Sinclair's work in that I am interested in seeing proof, at its genesis, as a dynamic, material intensity that has its own agenda rather than mathematics being centered exclusively within the subject/body. This work is novel in the field of mathematics education in that I continue de Freitas and Sinclair's research approach in the realm of mathematical proof where little (if any) work has been done. In the next section, I discuss how D&G's concepts such as intensity, body without organs, event, and immanence is connected to mathematics. I will then demonstrate these ideas in a story written by a professional mathematician found in the book *Living Proof: Stories of Resilience Along the Mathematical Journal*, in order to show the relevance of D&G's ideas in the teaching and learning of mathematical proof.

Proof as Intensive Production

In Deleuze's (1969/1990) *The Logic of Sense*, the author presents the notion of the *event* and initially aligns this idea with the static genesis. That is, the event as something incorporeal and transcendent. With the dynamic genesis, the event is no longer associated with something sterile. Rather, the event, is produced and not simply a "result" or a "condition." Consider an

example from Smith, Protevi, and Voss (2023) regarding a hurricane. Initially, prior to the formation of this particular weather event, there is a chaotic assemblage of varying wind speeds, temperatures, and moisture that may or may not become something else. It is in this sense that pre-hurricane weather patterns are asignifying as they do not necessarily imply the formation of an actual hurricane. However, the moment these elements synthesize in just the right manner, a hurricane results. The initial, chaotic, pre-hurricane mixture is the dynamic genesis of the actual hurricane that eventually finds its being. Unlike the Battle of Waterloo example previously discussed, the hurricane is connected to a material existence in its dynamic genesis (the pre-hurricane mixture). The "battle," however, in "The Battle of Waterloo" is merely a static expression or condition of its proposition and attribute of its state of affairs. "Expression," attribute," and "condition" are transcendent. The pre-hurricane mixture, however, is not since it is connected to and actively produces the corporeal event of the hurricane in the world of the material (not the transcendent). The "battle" is not produced but is merely an incorporeal, unproduced, effect of language and states of affairs.

For Deleuze (1969/1990) the dynamic genesis, which is intrinsically connected to the question of the body without organs and the produced event, is much more interesting than the static genesis and its unproduced forms of sense and event. The dynamic genesis denies the idea that meaning and truth are assumed, ready-made, transcendent, and something to take for granted. For D&G, truth itself is produced in becoming. How can proof be conceptualized in terms of this dynamic genesis? To explore this, I next analyze excerpts from an autobiographical story written by David Neel from the book *Living Proof: Stories of Resilience Along the Mathematical Journey*[3] published by the American Mathematical Society. In doing so, not only

[3] Excerpts reprinted with permission from the American Mathematical Society and their respective authors.

will I demonstrate that proof has a dynamic genesis steeped in asignifying materiality but I will also discuss how this genesis relates to the teaching and learning of proof as well. Neel's (2019) story describes an event near the author's completion of his doctoral dissertation in mathematics. While he had already accepted a job offer, he was still struggling to obtain some important results to complete his research. His supervisor, Ken, invited David to his summer home to finally put together the missing pieces. Neel (2019) writes:

> But by three days in, though we knew the problem more intimately, the long sessions at the whiteboard in his basement office had failed to produce the breakthrough. I felt claustrophobic, anxiety-ridden at the closeness. Each day, all day, we were together. Were we stuck? (p. 85)

So far, working in the traditional environment of the mathematician did not appear to get the pair where they wanted. I suggest that the "office" is really akin to the interiority of the mathematical subject. That is, the pair was trying to obtain their desired results via a static genesis in which mathematical meaning is actualized through the search for sense in the mathematical propositions and states of affairs they were studying and generating. However, Ken eventually suggests that they take a hike in a state park located close to his home where they would "Talk through the problem, aloud, feet in motion, see if we could sketch out the missing pieces" (Neel, 2019, p. 85). This suggestion is an important instructive move that Dave's dissertation supervisor makes. Continuing, Neel (2019) begins to describe the hike:

> The hike to lunch took closer to 45 minutes, but the weather sparkled and my flat feet didn't feel too bad. Most importantly, the shifting angles of the light, the mixed and faded shades of evergreen trees and grasses, the brown of mud and trail, the fact that we were

> watching our feet, and the sky, and the undergrowth, this all somehow focused our concentration into a pair of tiny sparks, dancing amongst dry tinder. (p. 85)

Notice Neel's (2019) words: "the shifting angles of the light, the mixed and faded shades of evergreen trees and grasses" (p. 85). The author is describing the subsequent hike as an assemblage of all of these elements, synthesizing in such a way that it "focused our concentration into a pair of tiny sparks, dancing amongst dry tinder" (p. 85). These elements are held together by an intensity that cannot be divided or broken. The hike, or in this case the "proof-hike," is this intensive assemblage. Neel (2019) continues: "I almost worried when we stopped in the clearing. We had no new insight yet, but something loomed, felt nearer somehow. We had stated the problem with more clarity, or with a more promising perspective" (p. 85). Observe Neel's (2019) emphasis here on a feeling of "closeness." What are they close to? They are close to the event that will actualize the proof they are currently pursuing. This is evidence that both David and his supervisor are now operating in terms of a dynamic genesis that is producing the event as opposed to the static genesis that is primarily concerned with an unproduced, incorporeal sense or meaning. Neel (2019) writes:

> "You know,'' Ken said, repacking his pack, "we could just go back, but it seems like we're on the right track. Do you want to hike in a bit deeper and keep talking?'' I ignored my aching fallen arches and nodded. Follow Ken's lead. He's the one who's been through this all before. (p. 85)

Notice how David's dissertation supervisor Ken facilitates the intensity of the proof-hike. Rather than dictating the actual mathematical ideas involved, Ken allows David to make the decision as to continue the hike or not. David then proceeds to "follow Ken's lead," that is, to follow the intensive flow that constitutes the proof-hike assemblage. This is a desire-laden construction

which does not seek to resolve a mathematical lack, that is, the incomplete results they are seeking, but rather seeks to reach the plane of immanence where proof is purely an affective endeavor rather than as a logical method for obtaining mathematical results that were not actualized previously. When Neel (2019) writes, concerning Ken, that "He's the one who's been through this all before" (p. 85) we might read this as "He's the one who's reached the plane of immanence before." When reaching the plane of immanence through proof, one engages with proof at its intensive, material, and asignifying genesis. It has its own agenda. As Deleuze and Guattari (1987) write concerning the BwO, "there is nothing to interpret" (p. 153). There is no incorporeal, transcendent meaning to obtain. One does not need to "understand" mathematics to reach the plane of immanence through proof. Rather, one simply needs to be open to proof's affect. That is, to experience proof as "an opening up of possibilities" (Rota, 1997, p. 191). Next, Neel (2019) begins to describe the pure affect that is involved with reaching the plane of immanence through proof:

> We reached another split in the path. We may have made a joke about graph theory. Two paths, but the shorter one was marked: "Path Closed for Maintenance." Still two options, really: forward or back. We do not have a map. Ken is pretty sure he remembers where this forward path emerges. It should be fine. Keep walking. Keep talking about combinatorics....It feels as if this has become some elemental struggle. And we're still, somehow, talking about combinatorics. (p. 86)

Note Neel's (2019) use of the phrase "elemental struggle." This further suggests that the work of proof is not all located in the writing and comprehending of formal mathematics or that the proof lies in the mathematical meanings and heuristics that our minds tend toward. Instead, it is wrapped up in the materiality of the body and its intensities. That is, in the proof assemblages

that form the dynamic genesis and create the potential and possibility for mathematical results. This “elemental struggle” is the proof assemblage and its active construction. In Neel’s (2019) story, the assemblage that forms is that of the “proof-hike” of which both David and his supervisor Ken are engaging with and that is akin to the pre-hurricane mixture discussed in Smith, Protevi, and Voss (2023). The proof-hike is an assemblage that synthesizes varying forces such as the earth, movement along a path, combinatorics, both David and Ken, etc. Also, observe that Neel (2019) is describing a kind of flow: “We reached a split in the path. We may have made a joke about graph theory,” “Still two options really: forward or back. We do not have a map,” “Keep walking. Keep talking about combinatorics” (p. 86). This is not a traditional proof that is being constructed. Rather, this is an intensity that is gradually shaping a proof BwO into something with the possibility for actualization into the axiomatic. At the level of immanence, proof is desire in itself rather than in terms of some kind of solution that has yet to be found. This is proof in itself and not in any signifying relationship to a theorem. Ken is currently teaching his student how to construct proof at the level of the dynamic genesis rather than the more axiomatic, static genesis that is typical in traditional classroom settings and where Ken and David, perhaps, were working while in the office space prior to the hike. Unlike the static genesis, David is becoming “worthy of the event” (Deleuze & Guttari, 1994, p. 160). This is not in any capitalistic sense, but in the sense that proof is a material production which must be carried out in order for the dynamic genesis of proof to actualize its axiomatic form. Being “worthy” of proof simply means to allow a material process to occur rather than working hard enough or being “intelligent” enough to solve a formal mathematical problem. What does this event, which the intensities of the proof-hike has been building towards, look like? Neel (2019) describes this:

> But now, we have it. It snapped into place. Now, we are sketching it out more fully, outlining it for each other, repeating, so that if only at least one of us can somehow walk or crawl free once more into that world outside this state park we can explain it to someone. (p. 86)

The “snapping into place” is the moment of actualization where David has become worthy of the proof-event. This event does not mean that the proof process is fully complete. Even Neel (2019) describes having to sketch out “more fully” the details of the argument. It seems, however, that they have achieved a sense of conviction in which the heuristic or “key idea” of the proof has been found and where the details just need to be ironed out. There is even a sense of wanting to share their solution publicly, which suggests they are close to obtaining the formal argument. This is what the dynamic genesis of proof looks like: deterritorialize the signifying elements of formal mathematics and allow the asignifying materiality of proof to take over; reach the plane of immanence and follow the intensities toward actualization.

The hike in Neel’s (2019) story is not a metaphor: The hike is the proof. However, it is not the literal hike that is the proof. Rather, it is the underlying intensities and assemblages that constitute the hike that also constitute the proof. To prove is to hike, but that does not necessarily imply that we should be taking students outdoors in order to reach the plane of immanence. Reaching the plane through proof requires proof-hikes not literal hikes. That is, it requires that students form assemblages.

There is one more important aspect of Neel’s (2019) story that carries significance. At the very end of Neel’s (2019) chapter, he writes:

> The moral is clear: care for one another, keep walking, do not despair. One other moral: Ken's example, his kindness and generosity. He was a model mentor and a good man. I had hoped and expected more years and many more chances to thank him. (p. 86)

Although David and his supervisor find the solution they are looking for by first deterritorializing on the plane of immanence and then reterritorializing or actualizing the assemblage they had constructed, Neel's (2019) final conclusion is not a purely mathematical one but an ethical one as well. He expresses what Roquet (2014) refers to as "cosmic subjectivity" or "a form of self-understanding drawn not through social frames, but by reflecting the self against the backdrop of the larger galaxy" (p. 124). In the case of Neel's (2019) story, the galaxy here is the plane of immanence that has the potential for both mathematical and ethical possibilities. This subjectivity is akin to Rota's (1997) view that proof should be "an opening up of possibilities" (p. 191). This subjectivity also corresponds to Stinson and Bullock's (2012) notion of a "praxis of uncertainty" where, invoking Freire, "speaking a true word and transforming the world are both left open to multiplicitous possibilities" (p. 49). David may have achieved conviction regarding the literal proof that he had been working on, but his supervisor Ken, at the same time, is successful in teaching David that, ultimately, proof is an ethic of uncertainty. This ethic, I believe, has the potential to undo these attitudes of certainty and, as Ernest (2021) puts it, "the ideology of purity" (p. 3146) that often pervades mathematics itself. Purism may exist at the level of a proof's actualization but it does not exist at its dynamic genesis. Proof is, to use Deleuze's terminology, a form of problematics in which exploration and possibility are primary to axiomatic certainty. Regarding problematics versus axiomatics, Smith (2006b) writes:

> In this way, the ontological status of the problem as such is detached from its solutions: in itself, the problem is a multiplicity of singularities, a nested field of directional vectors which define the 'virtual' trajectories of the curves in the solution, not all of which can be actualized.…the equations can define the virtual 'attractors' of the system (the intrinsic singularities toward which the trajectories will tend in the long-term), but they cannot say in advance which trajectory will be actualized (the equation cannot be solved), making accurate prediction impossible. A problem, in other words, has an objectively determined structure (virtuality), apart from its solutions (actuality). (pp. 161–162)

In relating the idea of problematics to mathematical proof, I want to suggest that proof also has this kind of virtuality. That is, proof, at its dynamic genesis, is not something in which a solution can be signified. Rather, it is a material assemblage, a hike in the case of Neel (2019), in which one revels in the affect of virtual possibility. In the case of proof, the "objectively determined structure (virtuality)" that is "apart from its solutions (actuality)" (Smith, 2006b, p. 162) involves the bottlenecks, critical points, and movements involved with the proof process. In the case of Neel (2019), these correspond with the moments of clarity and closeness, and the movements of the proof-hike. These are not part of the mathematical solution itself but are part of the virtual structure, the assemblage, that eventually actualizes some solution. Such virtual structure opens up possibilities for new ways of thinking. In the case of Neel (2019), while he was able to attain the result that he and his supervisor were looking for, something else was actualized in the process that was, perhaps, even more important than the actual solution: a "cosmic subjectivity" (Roquet, 2014) that allows for new ways for thinking about proof and mathematics. This appears to be consistent with Rota's (1997) view that "what an axiomatic presentation of a piece of mathematics conceals is at least as relevant to the understanding of mathematics as what an

axiomatic presentation pretends to state" (p. 192). Rota (1997) is arguing that proof may appear to be "definitive" at first glance but may actually hide the "true" result: "The error lies in assuming that a mathematical proof has been devised for the explicit purpose of proving what it purports to prove" (p. 190). In the case of Neel (2019), the "cosmic subjectivity" (Roquet, 2014) that Neel (2019) attains is the "true" result that is connected to proof's virtual structure. Rota's (1997) discussion is also consistent with Thurston's (1995) essay on mathematical progress. The author describes how he had so territorialized his particular field of mathematics that others refused to even participate in his research area. It was not until he began to withdraw from the field, deterritorializing it, that others began to discover new proofs, even for theorems that had already been proven, which then lead to the territorialization of new frontiers of mathematical thought. Deterritorialization, then, this reaching of the plane of immanence, must be reckoned with by mathematicians as evidenced by Rota (1997) and Thurston (1995). It is as much of a skill as territorializing mathematics via the more static axiomatic approach to mathematics. Neel's (2019) story demonstrates how Ken, David's dissertation advisor, engages in a kind of instruction of the virtual. By suggesting and taking the hike in the nearby state park, Ken actively teaches his student how to deterritorialize and how to reach the plane of immanence so that one can revel in the possibilities of proof, to feel its material contours, and to make contact with its affect. Neel (2019) describes the encountering of these material sensations when he states: "something loomed, felt nearer somehow. We had stated the problem with more clarity, or with a more promising perspective" (p. 85). This type of exploration then eventually leads to the production of the "snapping into place" that serves as the event for a likely solution. However, nothing signifies this event and there is no way to scientifically measure the relationship between a problem and its solution. The production of an event is an act of asignification, of becoming,

that can only occur within proof's dynamic genesis. In order to become a mathematician, that is, to become one who proves, one must learn to deterritorialize the mathematical self as well as the axiomatic method so that what remains is the problematics of the body without organs joining the plane of immanence. In the next section, I discuss some possibilities pertaining to the teaching and learning of proof at the undergraduate level that is based on Neel's (2019) story and the theoretical ideas developed in this chapter. I also provide some commentary on the notion of affect regarding our approach to mathematics education research.

Possibilities for the Teaching and Learning of Proof at the Undergraduate Level

Proof, at its genesis, is an asignifying intensity and not a stable identity or meaning. The genesis of proof can be accessed by reaching the plane of immanence via the formation of proof assemblages (e.g., the "proof-hike" described in the previous sections) or the seemingly arbitrary bringing together of heterogeneous forces that facilitates an intensity leading to the production of the proof event. This event actualizes the sense of a proof, leading, ultimately, to its state of affairs and its formal mathematical representation. However, the production of the proof event is a mysterious, asignifying one in that it is not causal in a scientific sense. A proof's becoming cannot be measured since it does not resemble that which it becomes. What we can do is utilize a philosophical methodology to open up proof's becoming (what I call *becoming-proof*) and to use concepts to further understand its asignifying affect. This is perhaps one way to summarize the implications of D&G's philosophies as they pertain to proof. How, then, does this relate to the teaching and learning of proof at the undergraduate level? I argue, based on Neel's (2019) story material analyzed previously, that in order to become one who proves, one must learn to reach the plane of immanence via the formation of these proof assemblages. With this in mind, I want to discuss several broad suggestions for how this might be done in the teaching and learning of

proof at the undergraduate level. Recall that a proof's productive genesis does not resemble the object of the production. It is important to remember this as we investigate these implications. Since proof, at its genesis, does not look like a signifying, axiomatic process where ideas are deduced in some logical or cognitive manner, the possibilities for the teaching and learning of proof based on D&G's ideas will also not resemble the axiomatic, logical, or cognitive sense-making aspect of proof as given in much of the mathematics education literature.

The role of the personal and the impersonal in the learning of proof.

From a Platonist perspective, proof is a process in which we discover or approximate the facts and theorems that constitute mathematics. There is an impersonal aspect of proof that, perhaps, denies the human component involved with mathematics and the teaching and learning of mathematics. By seeing proof as an assemblage, such an impersonal component of proof remains in the sense that assemblages have an existence outside of the mathematical subject. At the same time, these assemblages have the capability to integrate with the subject so that proof is no longer just an impersonal encounter. Regarding Delueze's view of immanence, Philp Goodchild said that "It's not the case that thought is within us, but thought is a kind of environment that we enter into and are already within" (Timeline Theological Videos, 2014, 9:09). This is how proof as becoming works as well: it is the process in which the mathematical subject becomes integrated with other material components to form an assemblage that leads to a potential for a mathematical idea, as well as other ideas, to actualize. This becoming is deeply personal and involves an intimacy with the environment, or assemblage, that one enters into when engaging with proof.

This is where the instructor plays an important role. Neel (2019) demonstrates that a close, personal connection with one's mentor allows for intensities to flow more easily within the

proof assemblages that are constructed in a mathematical encounter. This relationship is an immanent one and should give us pause in terms of how this dynamic may, or may not, be reflected in the typical undergraduate proof classroom. Do our classrooms allow for students to intimately experience the sort of cosmic subjectivity (Roquet, 2014) that appears to be present in Neel's (2019) experience? I would argue that the proof classroom is more likely to block such becomings, as it will usually promote a transcendent approach to the study of proof (instructor and textbook as primary arbiters of knowledge, focus on classroom chalkboard work, etc.) as opposed to the more immanent one that Dr. Neel's supervisor helped to facilitate (outdoors, close connection to nature and the mentor, etc.). The transcendent situates proof, mathematics, and the instructor as distant, less capable of assembling with the subject, and devoid of a sense of wonder, awe, humility, and soul. This is not to say that a traditional, lecture-style proof classroom is completely incapable of immanent moments or that these spaces cannot produce students who eventually develop such an immanent proof-sense, but rather that the platonism inherent in this more traditional environment renders such experiences less likely and more difficult to access.

The impersonal must be manifested immanently through an assemblage rather than through an approximation process that foregrounds the transcendent. In Neel's (2019) case, the assemblage of the mathematical material with the hike and the very personal relationship between Dr. Neel and his supervisor ensures that the emerging proof is an intensive, and therefore an immanent, production that not only results in a mathematical "solution" but also in new affective knowledge. While Dr. Neel's sense of the cosmic does not signify anything mathematical, it is, perhaps, a form of asignifying mathematical knowledge that is very personal, meaningful, and connected to the proof process. Therefore, the ability and the potential to be

affected is an important part of proof and mathematics. Without such a potential, there would be neither intensity nor becoming. In order for proof to "happen" there must be a dynamic genesis that begins with the affect of the asignfying body. These affects form the assemblages that allow for a proof to emerge in the first place. The construction of such assemblages allows the mathematical subject to be "worthy of the event" (Deleuze & Guattari, 1994, p. 160) or in this case, the proof event.

Just like in Neel's (2019) story, a good place to start in developing an immanent approach to proof instruction would be to emphasize closer, personal relationships between students and instructors, graduate teaching assistants, and other potential mentors. These relationship would help to foster and grow the proof assemblages that students are constructing in these learning environments by allowing intensities to pass that go beyond the transmission of axiomatic knowledge. Instructors have the capability to transmit feelings and interests that may not be as possible to communicate during a lecture session. For example, instructors may be more willing to open up about their own personal experiences with proof as a student or even in their own professional work as a mathematician. Even "small talk" that does not have a clear connection to mathematics might be enough for a particular kind of intensity to pass that contributes to the student's proof assemblages. In light of D&G's thinking, I believe that the closer students are to their instructor, the closer students will be to the concepts and ideas in which they are learning. If this connection is successfully established, then it is much easier to construct the assemblages that lead to further encounters with mathematical immanence as well as a greater potential for completed, axiomatic proof productions.

A new way to approach affect.

While there has been a fair amount of studies on affect in relation to mathematics education (e.g., McLeod, 1992), it "is still a relatively new endeavor" (Cai & Leikin, 2020, p. 288) and, according to Seldon, McKee, and Selden (2010), "often seen as separate from, but related to, cognition" (p. 200) encompassing aspects of the "passive mind" (p. 203) including beliefs, attitudes, emotions and values. Affect then, in the mathematics education literature, appears to be situated within one's consciousness. While affect (or intensity) does indeed function with the subject, it is also often operating external to it. Physical processes, objects, bodies, works of art, material conditions, etc. can all be seen as affects when utilizing the ideas of D&G. These conditions form the assemblages which individuals desire within and are a part. Proof, then, does not begin with the mind but, rather, resides within constructed material assemblages. In the example of Neel (2019), David's proof resides within the hike he and his supervisor take and while an axiomatic proof is being constructed during the hike, the genesis of his proof is not axiomatic. Rather, David and his supervisor Ken construct an assemblage which the axiomatic proof functions with and within. Here, an infinite number of intensities flow and enter into relation with each other including David himself, his supervisor Ken, their bodies, the earth, the paths that they follow, the lunches that they take with them, the mathematical material which they work from, their own thoughts and feelings, convictions and certainties (or lack thereof), as well as the axiomatic proof itself. This assemblage is proof at its true genesis (as opposed to only proof in the axiomatic). These elements are intensive, transmitting forces between one another in order to keep the assemblage together. The larger "the assemblage grows," (de Freitas & Sinclair, 2013, p. 464) the closer David and Ken are to their proof's

actualization. What might this way of understanding mathematical affect mean in relation to our research approaches?

As researchers, by seeing proof and affect in the manner described above, we obtain more analytical power. Power that we may not have accessed by seeing affect as merely something that originates within the mathematical subject. With D&G, everything is affect and contributes to the various assemblages we find ourselves thinking with and within. Stories about proof, then, are the proof assemblages that are found at the genesis of axiomatic proof itself. Aspects that might seem inconsequential now take on new roles as part of a proof assemblage. This research approach may even allow for new types of analyses with regards to classroom observations where seemingly arbitrary discussions, movements, and events now function necessarily within the proof assemblages that students construct. The actualization of these assemblages (whether via the successful resolution of a proof or any other type of problematic), results in something that may or may not signify something mathematical. In the case of Neel (2019), the resolution of the proof assemblage featured in that story resulted in a solution to a problem but also the actualization of a "cosmic subjectivity" (Roquet, 2014) that is not typically associated with mathematical proof. For Neel (2019), proof is associated with such subjectivity, in which affective possibility is now primary over the extraction of a mathematical surplus value. By seeing proof as an intensity, then, new possibilities for proof emerge that may not have been seen as relevant before.

Teaching philosophy and ethics to mathematics majors.

As can be seen from Neel's (2019) story, the dynamic genesis of proof involves a kind of "cosmic" realization which suggests that the axiomatic elements that the mathematical subject traditionally works with are only a part of a larger plane of immanence. Such a plane is affective,

intensive, and involve, as Hughes (2008) states citing Deleuze, “an ‘ethics’ of intensity” (p. 123). Such an ethics is not typically a part of an undergraduate mathematics program. In Ernest (2018), the author suggests that philosophy should be taught to our mathematics students, indicating that

> Since mathematics is the essence of instrumental reason, with its focus on means to ends and not on underlying values, and its procedures require standardization, routinization, and dehumanization, the concomitant erasure of ethics is no surprise. Thus a training in mathematical thinking, when misapplied beyond its own area of validity to the social domain, is potentially damaging and harmful. (p. 198)

I agree with Ernest’s (2018) views here, and I think this is particularly the case with mathematical proof as there is an illusion that proof produces absolute knowledge that is, when done “correctly,” incapable of error. However, we know that proof cannot secure knowledge absolutely. As Epstein and Levy (1995) noted:

> When an ordinary mathematical theorem is published, we do not require it to have a formal, mechanically verifiable proof. That would be too long and incomprehensible and may, in any case, be unattainable because of the world’s limited amount of paper, brain cells, computer memory, whatever. (p. 673)

That is, a typical proof, while hopefully thoroughly vetted, is never a guarantee of correctness and certainty. As Epsetein and Levy (1995) point out, unintentional errors in published proofs can go unnoticed for quite a long time. Therefore, the idea that mathematical proof is a vehicle for objective facts can only go so far. Furthermore, Raman (2003) suggests that expert mathematicians utilize informal heuristics known as “key ideas,” to transition their private understandings of a proof toward a more rigorous, public form. That is, experts have ways of comprehending that do not always coincide with formal, complete mathematical knowledge.

This further suggests that helping students to mature their epistemology and beliefs about proof and mathematics may assist in their understandings on the different approaches and strategies to obtaining formal proof presentations. It may also help students to see that proof is ultimately not value-free and, therefore, disconnected from ethics and possibility. Simply having a discussion with students concerning questions such as “Does proof secure absolute knowledge?” or “In what ways are pictures relevant to the proof process?” may help to undo any beliefs and dispositions that support any overly-strict view of proof as something that only deals in absolute certainty, perfection, correctness, and rigor. By exploring philosophy and ethics within the context of mathematics, instructors can help students to gain an intensive understanding of mathematics. This is why the “proof-hike” assemblage from Neel’s (2019) story is so important. Ken was able to show Dr. Neel that proof has a genesis not in the axiomatic but in the affective. Ken was teaching what Hughes (2008) calls, citing Deleuze, “an ‘ethics’ of intensity” (p. 123). Proof no longer has a static genesis in the axiomatic, in the absolute, and in the transcendent but now has a dynamic genesis in the invisible, unrecognizable world of becoming.

Not only can these discussions foreground topics related to different strategies and approaches to the proof process, but they can center on issues surrounding the ethics of mathematical “purism” and Eurocentrism as discussed by Ernest (2021):

> Alongside and intellectually justifying European empire building and conquests in the Southern, Western and Eastern continents there has been a growing Eurocentrism, the racist bias that claims that the European ‘mind’ and its cultural products are superior to those of other peoples and races. Against this backdrop it is not surprising that that mathematics has been seen as the product of European mathematicians. (p. 3150)

Discussions pertaining to these sentiments could help to undo misconceptions and harmful views, particular for mathematics majors who are beginning to learn and engage with proof and who are, perhaps, closest to the mathematical purism discussed in Ernest (2021). In summary, these discussions concerning epistemology, ethics, purism, eurocentrism, and possibility can serve as the basis for the formation of assemblages which allow for the reaching of a cosmic, intensive plane of immanence from which proof and mathematics emerge. Such an approach has the potential for the decolonization of proof as something that prizes the European way of knowing over other forms of knowledge. In this deterritorialized context, proof becomes that of the asignifying body without organs in that it is now “an opening up of possibilities” (Rota, 1997, p. 191) with a dynamic genesis as opposed to a static genesis that merely signifies or adequates a mathematical statement with its state of affairs via logical argumentation. As Neel (2019) learned during his encounter with the “proof-hike” assemblage, proof at its dynamic genesis is no longer about truth but about the complicated manner in which notions of truth are produced in the world, the affects and consequences of such a production, and the way this production can be reconceptualized. Such knowledge will allow our students to understand that proof and mathematics are not absolute ways of knowing (Ernest, 2021) but, rather, are cultural, political, and value-laden rhizomes.

These discussions are dangerous in that they deconstruct notions of truth and logic and, therefore, perhaps, threaten the very foundations of mathematics itself. Why study mathematics at all if truth is no longer privileged? However, I argue that this way of knowing, while asignifying, is still mathematical. That is, it still constitutes the dynamic genesis of proof itself and, therefore, to deny or avoid this dimension of proof in our instruction would be to provide an incomplete understanding of mathematics. Neel’s (2019) story demonstrates that this dynamic

genesis involves asignifying forms of knowing that is not typically associated with mathematics (e.g., the hike as an intensive dimension of proof itself) and, yet, is connected to the axiomatic proof process. Currently, there isn't a clear space for our students to explore this way of mathematical knowing which is why some dedicated time to the philosophy, ethics, and the dynamic genesis of mathematics should be prioritized.

Emphasize the problematics of proof rather than just the axiomatics.

As indicated in Rota (1997), proof is more about "an opening up of possibilities" (p. 191) rather than deriving a stable theorem result. Smith (2006b) notes that "the ontological status of the problem as such is detached from its solutions" (p. 161). Therefore, by focusing too much on proof as an "answer," we fail to see proof as something in itself, as difference in itself, and as something more than a means of mathematical signification. One way to have students participate in the world of problematics is to stress problems over the retrieval of solutions. There is fair amount of research in the area of problem posing in mathematics education (see, for example, reviews in Cai and Leikin, 2020 and Weber and Leiken, 2016). Concerning problems, de Freitas (2013b) states:

> The literature on problem-posing in mathematics education consistently produces studies of rich classroom interaction where students are invited to experiment with diagrams and produce conjectures about shapes, numbers and other 'mathematical entities' and relationships. The work on problem-posing resonates with the Deleuzian argument of this paper, suggesting that educators should treat problems as sites of creativity and invention, and less as that which must be resolved to make way for the solution. Despite the insights of this research, however, and despite the evidence that mathematicians themselves

> operate in this way, mathematics classrooms remain sites of compliance where regimes of truth enforce an axiomatic image of learning. (p. 594)

Like problem posing, where obtaining solutions is not the main goal, I suggest that taking a Deleuzian approach to proof involving problematics would entail a focus on identifying key features of a mathematical scenario while avoiding any expectation of a resolution. What would this look like in a proof course? I suggest focusing on easily graspable scenarios that simultaneously offer rich avenues for exploration such as the Königsberg bridges problem. This scenario involves a group of islands separated by rivers but connected in a variety of ways by a number of bridges. The problem is this: given a particular configuration of islands and bridges, is it possible to traverse each bridge exactly once? To engage with this problem in a manner consistent with Deleuzian problematics, an instructor may not even begin with the problem statement. Rather, the instructor might simply present the configuration of islands and bridges and ask the students to identify interesting features and questions from the picture. For example, students may find it interesting that some islands have more bridges adjacent to it than others or some may note that it is possible to walk to each island in a cyclic fashion by utilizing some bridges but ignoring others.

These discussions may eventually lead to more interesting problems such as the possibility of traversing each bridge exactly once. When the student or students decide on an interesting direction to pursue, further problematization may commence. In what ways is the number of bridges adjacent to each island significant? Does parity matter? With further prompting and exploration, students will hopefully see that at least some of the islands must have an even number of bridges connected to it since, if I enter an island via a bridge, I must also be able to leave it without using a previously traversed bridge. What is the significance of the

starting point (or island) and the ending point (or island) of our walk? Does it matter if I begin and end the walk on the same island? If I begin and end the walk on the same island, it would seem that the number of bridges incident to this island would also have to be even since, again, I need to be able to exit and enter the island without crossing a bridge more than once. The above is not a formal, rigorous proof but rather an identification of, in Smith's (2006b) words, "the ideal accidents and events that condition the problem and form the cases that resolve it" (p. 145). The goal here is not so much the resolution but the exploration of the problem's conditions and possibilities. I would argue that the ability to engage with this problematic side of proof is a skill in and of itself in addition to the more traditional, axiomatic approach to the proof process where "a deduction moves from axioms to the theorems that are derived from it" (Smith, 2006b, p. 145).

The main idea concerning the issue of the problematics of proof is that of the assemblage. When students are exploring a problem for its important qualities and properties, rather than simply finding a solution, what students are actually doing is constructing an assemblage and desiring from within it. In an interview with Clair Parnet, Deleuze states "I never desire some thing all by itself, I don't desire an aggregate either, I desire *from within* an aggregate….To desire is to construct an assemblage, to construct an aggregate" (The Deleuze Seminars, 2022, on desire, https://deleuze.cla.purdue.edu/lecture/lecture-recording-1-f/). This is precisely what Neel (2019) and his supervisor do via the "proof-hike" construction. By not exclusively focusing on the actual problem solution, like they were when they were indoors, prior to the hike, the problem, and therefore the proof process itself, becomes an intensity, difference in itself, rather than an attempt to imitate a mathematical ideal that is yet to be realized. The more "the assemblage grows" (de Freitas & Sinclair, 2013, p. 464), the more likely that the students will

encounter the event that actualizes a solution and, therefore, a theorem. In the case of the Königsberg bridges problem, the more features students begin to observe and map out, the more likely they will realize that the parity regarding the number of bridges incident to an island is important. If the number of bridges incident to an island is an even number, then this allows the possibility to move in and out of the island without crossing any bridge incident to the island more than once. However, if the number of bridges incident to an island is odd, then at some point I will have to either avoid crossing all bridges incident to that island or cross one of the bridges twice. Observing these conditions are important but, in and of themselves, they are not solutions to a problem. Rather, they form the assemblage that may create the potential for some possible resolution.

What makes this approach interesting is that this assemblage, unlike the example just discussed, does not need to consist of only mathematically signifying components. As was seen with Neel's (2019) story, David's assemblage consisted of his supervisor and himself, the mathematical material, the hike, and the earth. It is from within this assemblage that the hike and the actual axiomatic proof share in the same intensity and where David becomes "worthy of the event" (Deleuze & Guattari, 1991/1994, p. 160). I would suggest, based on Neel's (2019) story, that we encourage our students to grow their proof assemblages as far as possible. This would mean allowing them to bring in interdisciplinary and applied ideas into these assemblages. For example, perhaps instructors should provide students with opportunities to investigate the historical implications of Euler's Königsberg bridges problem which paved the way for the study of a very important field of mathematics: topology (Adams & Franzosa, 2008). In their investigations of the problematics of Euler's bridges problem, what qualities concerning this scenario proved to be of historical importance in advancing mathematics as a whole? By

assembling the mathematical content with the historical context, the study of proof becomes a more intensive one. In the words of Deleuze and Guattari (1991/1994), there is a "laying out of a plane" (p. 36), that is, a plane of immanence. By constructing proof assemblages by bringing in components that are in addition to the mathematically-signifying problem, students grow the assemblage further and a desire begins to flow. There is no object of this desire. That is, there is no singular, clear solution that one gravitates toward when constructing these proof assemblages. Rather, the desire here is asignifying, a pure proliferation of affect.

According to this theory of problematics, there is a point when this assemblage reaches a kind of "critical mass" and the problem is resolved "by the *intrinsic* conditions of the problem itself" (Smith, 2006b, p. 160). For Neel (2019), the intensities generated by a problem or proof assemblage eventually results in a "snapping into place" in which there is a sort of "accidental" resolution. There are many types of problems, involving advanced mathematical ideas that are proof-worthy, that lend themselves to exploration and mapping rather than resolution. The study of musical chords and scales, for example, often lead to problems regarding counting. How many 3-note chords are there? How many 4-note chords are there? Discussions about mathematical combinations and the binomial coefficient could help support students' assemblage production with regards to this problem of chord counting. Further investigation into the musical problem of chords would reveal that musicians also care about set classes of chords which is a form of equivalence class. All the major and minor chords together, for example, form a particular set class that are related via transposition and inversion, two types of musical operations. This sort of investigation would lead students into learning about dihedral groups and how to count equivalences classes. Again, like the Königsberg bridges problem, there is a laying out of a plane in which the musical problem of chord counting opens up an entire realm of mathematical

possibilities to explore and identify. It is this opening up of a problem's possibilities, rather than simply having the goal to solve it, that is a skill in and of itself and is the primary concern of problematics. When laying out these planes of immanence, one can no longer ignore the ethics of possibility in exchange for a platonic ethics of duty or morals that emphasizes theorems as the ultimate goal of mathematics.

Desiring proof.

In Neel's (2019) story, the author details his anxiety concerning his difficulty with the mathematical results he is trying to obtain. This difficulty is marked by a sort of existential crisis in which he questions himself and his abilities, his future, and perhaps, by extension, proof itself. This leads to a new approach to proof, one that is governed by bodily intensity and a reaching of a plane of immanence rather than putting one's faith in the Western ideals of the axiomatic method that helps to support a capitalistic discipline with little connection to ethics. In a way, Neel's (2019) feelings resonate with the skepticism (of institutions, science, etc.) of the post-structuralist thinkers who began to emerge in the latter half of the 20th century. The hike that David takes is an assemblage in which he enters into a relationship with the mathematical material, the earth, and his mentor. With regards to proof, then, the body takes on a more fundamental role than the axiomatic method that is traditionally associated with proving. Proof becomes a desire rooted in the affect of the body and the assemblages it enters into relation with rather than a desire characterized by a longing for a theorem that is yet to be proven. This desiring-proof forms the ungrounded foundation for the deterritorialization of axiomatic proof. Proof is no longer what it once was. The mathematical subject, rather than engaging in a capitalistic ethic of mathematical profit in the form of a theorem, expresses a longing, a desiring,

or as Hughes (2008) states citing Deleuze, “an ‘ethics’ of intensity” (p. 123) and in Dr. Neel’s case, a cosmic subjectivity.

The instructor also plays a significant role in this process. Notice how David’s supervisor Ken suggests the hike, a movement away from the territorialized environment of the office space, which ultimately leads to a wandering on the plane of immanence. This wandering is also a problematic and a material exploration of its conditions that allow for a solution to emerge via a bodily desire rather than a desire associated with a lack of a theorem. Ken is an experienced mathematician and, according to Neel (2019), “the one who’s been through this all before” (p. 85). While proof as bodily desire is often not considered in proof and mathematics education more broadly, I want to suggest that in order to become one who proves, one must learn how to desire proof. Therefore, instructors such as Ken have more to offer their students other than axiomatic knowledge. One does not need to “understand” mathematics, in a signifying sense, in order to reach the plane of immanence. Rather, via the asignifying body, instructors can teach students the most fundamental aspect of proving: *to desire proof* in the first place and to desire in and of itself on its own terms rather than in terms of some missing axiomatic knowledge. We must find ways for mathematicians to approach their instruction so that this important kind of proof knowledge is shared as intensive, affective, and with the potential to form assemblages that open up the possible rather than closing on a static result.

Finally, by emphasizing the asignifying body, not only do we deterritorialize proof but we also deterritorialize the mathematical subject. Axiomatic proof naturally points to theorems and facts. If it does not, and one is therefore unable to accumulate, as Thurston (1995) puts it, “theorem-credits,” then it is rejected as deficient. Perhaps this is also communicated to the students who are first learning about the proof process. As Deleuze and Guattari (1980/1987) put

it: “You will be organized, you will be an organism, you will articulate your body–otherwise you’re just depraved. You will be signifier and signified, interpreter and interpreted–otherwise you’re just a deviant” (p. 159). Perhaps this was somehow communicated to Dr. Neel when he was a graduate student struggling with his dissertation. Neel’s (2019) feelings of deficiency, prior to the deterritorialization that occurred during the hike, seems to be suggestive of a kind of rejection. Perhaps this is what axiomatic proof communicates to our students. Unless we are able to also teach them the side of proof that is disorganized, the “side facing *a body without organs*” (Deleuze and Guattari, 1980/1987, p. 4), then this will likely continue to be the case in our university mathematics programs. Therefore, by finding ways to disrupt the Eurocentric, signifying side of proof by forming BwOs and reaching the plane of immanence, we can hopefully discover approaches that will allow for individuals who may feel that they don’t belong in these environments, to feel more welcome and to feel that they don’t need to signify an any particular way in order to be successful. By deterritorializing proof, we also deterritorialize the mathematical subject and the expectations that are associated with that subject, so that advanced mathematics classes can become a site where everyone can feel connected to the proof process rather than a privileged few.

Conclusions

In this chapter I have discussed Deleuze and Guattari’s philosophy of immanence, one that entails the intensity of the body as an asignifying becoming. This becoming does not resemble anything concrete or stable mathematically. It does not resemble that of axiomatic proof which is the way mathematicians would typically conceptualize this activity. Proof as a BwO does not privilege meaning, consciousness, the mathematical subject, or mathematical transcendents or objects. Rather, it is an intensity, or desire, that does not tend toward a theorem

or fact but is an assemblage consisting of material components (feelings, affects, activities, etc.) in which an axiomatic proof has the potential to emerge from. This immanent conceptualization has implications for instruction that will not resemble that of typical prescriptions found in mathematics education research. These possibilities involve developing intimacy with the impersonal material elements of an assemblage, supporting the construction of such assemblages via the study of ethics, epistemology, and ontology of mathematics, engaging with the problematics of proof by focusing on exploring problems rather than solving them, and, finally, teaching students how to desire proof in an asignifying sense. All of these aspects have the potential to deterritorialize the mathematical body so that proof is no longer the pursuit of mathematical capital in the form of a theorem but a desire with no subject, object, barriers, or restrictions that keep our students from reaching the plane of immanence.

Conclusions

In the previous chapters, I have considered how proof can have an existence beyond the axiomatic and the signifying. Proof does not simply have a *static* genesis, that is, it is not merely the adequation of a theorem or fact with its justification or proof via some sense that is obtained through constructive or cognitive processes that is consistent with thought as partaking in logic, reasoning, and meaning-making. Rather, the genesis of proof also has a *dynamic* genesis, one that does not privilege the mind and meaning but intensity, the body, affective desire, and becoming. These aspects are part of the genesis of mathematics without signifying or resembling anything particularly mathematical.

In Chapter 2, I discussed the fan community of the electronic music duo Boards of Canada (BoC) and their interest in creating their own mixtapes and music videos. In one particular YouTube fan video (https://www.youtube.com/watch?v=UsOZ1NKInsE) posted by the channel Eugene One (2012), we are shown clips of various birds taking off in flight set to the track "Slow this Bird Down" by Boards of Canada (2005). Each clip is presented in slow motion, so that the intricacies of the wing movements can be seen in exquisite detail, all set to the audio of BoC complete with a solemn electronic choir in the lower registers. These images demonstrate the intensive assemblage that birds must construct in order for flight to be possible (e.g., wings in connection with space, the air, perhaps certain windspeeds, etc.) and that while our understanding of science and biology may help to explain the process of avian flight, there is still a mysterious affect, an immanence, that presides over such processes. Artist Paul Uhlmann, in the journal *Unlikely*, wrote

> birds have occupied a unique place within my art as creatures which conceptually open portals to immanence. In contrast, within art history, birds often carry a symbolic

function of transcendence–of being creatures outside of our lives, linking heaven and earth as messengers from God….however an overriding understanding is that they are very much part of the organic whole of Nature, the stuff of matter, which will ultimately lose this ability to ascend to the sky and decay into the earth (Uhlmann, n.d., para. 1)

Near the end of this piece, Uhlmann (n.d.) writes "humanity is deeply intertwined with nature, so what we do to the environment we do to ourselves" (para. 4). From this, we can see that humanity is also a part of the assemblage that birds utilize in order to take flight.

I want to suggest that, like the example discussed in the previous paragraph, proof also has traditionally been seen in terms of transcendence, as something ideal to be approximated or discovered, and as something distant that somehow presides over the structure of thought itself. However, examples described in this dissertation, such as Neel's (2019) story, suggest something very different: that proof is also in assemblage with nature, the Earth, and with the affective body or body without organs. It is connected to humanity so that, in rewording Uhlmann's (n.d.) thoughts, "what we do to or with proof, we do to ourselves." Neel's (2019) affective sentiments at the end of his proof story demonstrates this:

> The moral is clear: care for one another, keep walking, do not despair. One other moral: Ken's example, his kindness and generosity. He was a model mentor and a good man. I had hoped and expected more years and many more chances to thank him. (p. 86)

Neel's (2019) experience with the "proof-hike" assemblage produced an intensity that acknowledges the connection between axiomatic proof and the world. This affect is a sense that the proof process has an existence beyond the axiomatic. It exists on a different plane but not one that is separate from us, locked away in the transcendent. Rather, proof is present

with us, here and now, in immanence. If we agree that this is indeed the case, then this has consequences for the teaching and learning of proof. Instruction must be modified in ways so that students can feel connected to the immanence of proof. Classes that are focused on traditional lecture, where the instructor serves as the arbiter, or proof sense that exists between the student and the transcendent facts that constitute mathematics, will not as easily produce the sort of intensity that leads to becoming. As Neel (2019) demonstrates, reaching the plane of immanence is a very personal experience and students need to feel connected to someone who not only serves as a mentor for the navigation of the axiomatic method but also shows students how to construct affective assemblages that constitute the genesis of an axiomatic proof. By suggesting and taking the hike with his student, Dr. Neel's supervisor demonstrated that proof is connected to the world and not just the mind. He knew that the proof they were working on shared in the same intensity as the hike and that the two assembled to form the possibility for a new theorem. This "proof-hike" assemblage, I argue, is what constitutes mathematics and proof on the plane of immanence. It is "proof in itself" and not in primary relation to transcendent theorems and lemmas.

This dissertation only begins to explore the immanence of proof and there are further avenues for investigation into this area. Further analysis of story data would be useful to explore a broader scope of how proof functions and assembles with the world. Collecting written story material from undergraduate mathematics students would further assist researchers in understanding what the genesis of proof looks like at the level of undergraduate mathematics majors. How would these stories, written by undergraduate mathematics majors, compare with stories written by experts or even those who may not be formally studying mathematics? What might these say about how individuals learn about the

proof process? Another variation of this sort of pursuit might involve having individuals write fictional stories connected to proof and the proof process and exploring what these fictional accounts say about proof as “an opening up of possibilities” (Rota, 1997, p. 191). Stories are ideal here because they show that proof, like anything else, are in rhizomatic connection with other events, modes of thought, and materials. They show that even the most transcendent of identities are always in the process of assembling on the plane of immanence. Another potential avenue for further investigation might involve observing the classroom events that occur in a undergraduate proof course. What intensities exist here and what might happen that either encourages or blocks these intensities and becomings in these environments? How does the lecture format for these courses support the stable, axiomatic proof process that dominates these courses?

In addition to further empirical avenues as described above, there are also many other concepts that D&G explore in their various writings that were not addressed in this dissertation. For example, D&G’s ideas might be utilized more explicitly to draw attention to the Eurocentric aspects of mathematics, proof, and the ways that these are sustained. Related to this, In *A Thousand Plateaus*, Deleuze and Guattari (1980/1987) explore the concept of *faciality* which, in part, consists of semiotic systems that support whiteness and all of the cultural, religious, and moral processes that maintain it at the exclusion of those who may not signify in such systems. For Deleuze and Guattari (1980/1987), the face “is White man himself” (p. 176) and everything from television, painting, and even the clothes we wear are meant to center the face, to accent it, to produce it, and to have it dominate over the body and head. In this sense, faciality is an invisible process which has taken place right in front of us, across all of Western history. However, prior to colonization, there were no faces, only

bodies and heads: Deleuze and Guattari (1980/1987) write: "'Primitives' may have the most human of heads, the most beautiful and most spiritual, but they have no face and need none" (p. 176). How might mathematical proof be produced by faciality processes? That is, how might proof be an example of a "face?" Deleuze and Guattari (1980/1987) also associate the face with walls that signify despotic regimes and leaders, and it doesn't take long to notice, after looking at a written proof in a textbook or on a chalkboard during an abstract algebra class, that walls are a part of mathematics. How did this wall (or face) form? Perhaps the axiomatic method is a faciality process that somehow produced the style of instruction that emphasizes blackboards and the heavy use of lecturing? Perhaps the entire proof classroom is a face that is the result of mathematics itself with its emphasis on symbols and rules that support a regime that excludes those who do not signify within such Eurocentric systems. Perhaps, in writing out proofs on a board, we are also, without even realizing it, helping to produce this face? Future theoretical investigation might involve plugging in this concept of faciality with proof and seeing how they function. Historical investigations of mathematics and mathematicians may bring to light the faciality processes that have produced axiomatic proof and given it its form in the present. Concerning student learning, how might faciality processes manifest in mathematics students' experiences, in the classroom, and in students' textbooks? How do these processes function to exclude those who do not conform or signify in traditional university proof settings? This theoretical lens could provide helpful insight into why certain university students may struggle with proof and potentially help

investigators understand ways proof as a face can be, in the words of Deleuze and Guattari (1980/1987), "dismantled."

Finally, I want to conclude by providing some final thoughts about this work and how it has changed me as a scholar. I became interested in studying mathematics because of my fascination with the proof process. While I found more computational-based courses involving calculus, linear algebra, etc. interesting and challenging, I believe, at the time, I had an intuitive sense that this was not "really" what mathematics was. What I wanted, or desired, was to prove theorems and secure a kind of absolute knowledge. I felt close to "something" as an undergraduate taking proof classes much in the same way that Neel (2019) and his supervisor sensed that "something loomed, felt nearer somehow" (p. 85) during their hike. While part of this may indeed be a feeling of closeness to an "answer," I also believe this closeness was, at the same time, a reaching toward the plane of immanence, a plane where the privilege of the mind falls away.

What is left of mathematics if there are no transcendent minds, objects, and theorems to discover and to be made primary? Deleuze (1995/1997) writes: "It is sheer power, utter beatitude" (p. 4). Yes, mathematics has a transcendent dimension to it. How could it not? Mathematicians are constantly working, through the proof process, to approximate, signify, and imitate this part of thought. However, it also has a different side that exists simultaneously with the transcendent. This side is invisible and *cannot* be imitated. We can reach for it, begin to feel its contours, but it will always be out of our grasp. Unlike the world of transcendence, which is separate from us, the plane of immanence is *here with us now, in the present* and mathematicians are drawing from it constantly and mostly unknowingly. This is why it is difficult to teach this side of proof and mathematics. On the plane of immanence,

proof is only intensity and affect. There is nothing stable. To conceptualize proof on the plane of immanence is to no longer to make theorems and lemmas priority and to give up the privilege these stable identities hold. However, mathematicians and students of mathematics do not just experience the transcendent *or* the immanent. They are always working with both and in between these two worlds.

This leads me to another important conclusion about proof: the "line" between immanence and transcendence can be infinitesimally thin. For example, Deleuze and Guattari (1991/1994), write:

> *we want to think transcendence within the immanent, and it is from immanence that a breach is expected.…*Transcendence enters as soon as movement of the infinite is stopped. It takes advantage of the interruption to reemerge, revive, and spring forth again (p. 47).

As discussed earlier, birds are a good example of such a tension between immanence and transcendence. Uhlmann (n.d.) identifies them as symbols of the transcendent while at the same time seeing them "as creatures which conceptually open portals to immanence" (para. 1) and "very much part of the organic whole of Nature, the stuff of matter" (para. 1). Proof also engages in this tension as well. Logic, cognitive processes, ideal objects, the notion of the mathematician, theorems, and mathematical "facts" all attempt to signify and revive the transcendent and overtake the immanent. It is in this sense that the mathematically transcendent can easily be "seen." However, the plane of immanence is invisible. It operates, often without credit, in all aspects of the proof process and always at a greater level than anything transcendent. In fact, Deleuze and Guattari (1991/1994) state that only "When

immanence is no longer immanent to something other than itself it is possible to speak of a plane of immanence" (p. 47).

One does not need to literally go on a hike and experience the outdoors to get close to the immanence of mathematical proof. It can be reached anywhere, from the office of the lone mathematician to the classroom where the chalkboard takes on a face-like dominance. However, like the bird that forms an assemblage with the world when taking off in flight, something must be constructed, and perhaps, at the same time, deterritorialized, in order for one to "reach the plane of immanence" through proof. This is why I discuss various instructional-related possibilities in the previous chapter. These possibilities are attempts to reach out to immanence and to make contact with this plane that has always and will always preside over all of mathematics. The same is true for the research process in mathematics education, which often takes on a similarly transcendent posture by placing a strong emphasis on scientific empiricism, "data," and stable conclusions. For me, the most enlightening moments of my research process was not only reading D&G and the philosophical literature associated with this philosophy, but also listening to Boards of Canada, investigating video game speedruns, and understanding that these things could form affective assemblages with proof and the proof process much in the same way that the hike did in Neel's (2019) story. While these aspects do not seem to signify anything mathematical, they can become part of the research process and inform our thinking about a multitude of ideas. As long as we acknowledge the role of the immanent, they can become part of the assemblage of mathematics education research.

The role of asignification, both in the teaching and learning of mathematics as well as in the research process itself, suggests that mathematics, at the level of immanence, is not a

matter of platonic imitation in which one approximates transcendent facts and theorems that preside over mathematics itself. Such a transcendent conception is signifying and forces students to conform to a Eurocentric standard rather than accepting and celebrating differences and multiplicity of thought and experience. As illustrated in Neel (2019), proof's plane of immanence allows for a diversity of thought that axiomatics cannot facilitate as easily. The "cosmic subjectivity" (Roquet, 2014) that is produced during Neel's (2019) "proof-hike" changes how we understand mathematics and ourselves. We do not leave the plane of immanence with a capitalistic mode of thought that signifies results, answers, and solutions. Rather, we leave with a sense of wonder, awe, and compassion for others. I believe that this lays the groundwork for an approach to mathematics that is democratic and welcoming of all students and their diverse backgrounds and experiences. This is why conceptualizing proof and mathematics immanently is important as it allows for us to rethink mathematics from its foundations so that we can participate in a more just society where everyone can connect with mathematics and the proof process.

Regarding the plane of immanence, Deleuze and Guattari (1991/1994) write:

> Perhaps this is the supreme act of philosophy: not so much to think *THE* plane of immanence as to show that it is there, unthought in every plane, and to think it in this way as the outside and inside of thought, as the not-external outside and the not-internal inside–that which cannot be thought and yet must be thought (pp. 59–60).

Hopefully, in this dissertation, I was able to begin to "show that it is there" in the work of the mathematician and beyond while exploring ways to reach it in the teaching and learning of proof.

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